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W-algebras Associated to Truncated Current Lie Algebras

Thèse

Xiao He

Doctorat en mathématiques
Philosophiæ doctor (Ph.D.)

Québec, Canada

Résumé

Étant donné une algèbre de Lie $\mathfrak{g}$ semi-simple de dimension finie et un élément nilpotent non nul $e \in \mathfrak{g}$, on peut construire plusieurs algèbres-W associées à $(\mathfrak{g}, e)$. Parmi eux, l'algèbre-W affine est une algèbre vertex qui peut être réalisée comme une cohomologie semi-infinie d'une sous-algèbre nilpotente de $\tilde{\mathfrak{g}}$, où $\tilde{\mathfrak{g}}$ est l'algèbre de Kac-Moody associée à $\mathfrak{g}$. L'algèbre-W finie est l'algèbre de Zhu de l'algèbre-W affine. Dans les constructions des algèbres-W, une forme bilinéaire non dégénérée invariante et une bonne $\mathbb{Z}$-graduation de $\mathfrak{g}$ jouent des rôles essentiels. Les algèbres de courants tronqués associées à $\mathfrak{g}$ sont des quotients de l'algèbre de courants $\mathfrak{g} \otimes \mathbb{C}[t]$. On peut montrer que: (1) des formes bilinéaires non dégénérées invariantes existent sur des algèbres de courants tronqués; (2) une bonne $\mathbb{Z}$-graduation de $\mathfrak{g}$ induit des bonnes $\mathbb{Z}$-graduations des algèbres de courants tronqués. Alors, les constructions des algèbres-W fonctionnent bien dans le cas des algèbres de courants tronqués.

Les résultats de cette thèse sont les suivants. Premièrement, nous introduisons les algèbres-W finies et affines associées aux algèbres de courants tronqués et nous généralisons certaines propriétés des algèbres-W associées aux algèbres de Lie semi-simples. Deuxièmement, nous developpons une version ajustée de la cohomologie semi-infinie, ce qui nous permet de définir les algèbres-W affines associées à des éléments nilpotents généraux d'une façon uniforme. À la fin, nous prouvons que les algèbres de Zhu de niveaux plus hauts d'une algèbre vertex conforme sont toutes isomorphes à des sous-quotients de son algèbre enveloppante universelle.

Abstract

Given a finite-dimensional semi-simple Lie algebra $\mathfrak{g}$ and a non-zero nilpotent element $e \in \mathfrak{g}$, one can construct various W-algebras associated to $(\mathfrak{g}, e)$. Among them, the affine W-algebra is a vertex algebra which can be realized through semi-infinite cohomology, and the finite W-algebra is the Zhu algebra of the affine W-algebra. In the constructions of W-algebras, a non-degenerate invariant bilinear form and a good $\mathbb{Z}$-grading of $\mathfrak{g}$ play essential roles. Truncated current Lie algebras associated to $\mathfrak{g}$ are quotients of the current Lie algebra $\mathfrak{g} \otimes \mathbb{C}[t]$. One can show that non-degenerate invariant bilinear forms exist on truncated current Lie algebras and a good $\mathbb{Z}$-grading of $\mathfrak{g}$ induces good $\mathbb{Z}$-gradings of truncated current Lie algebras. The constructions of W-algebras can thus be adapted to the setting of truncated current Lie algebras.

The main results of this thesis are as follows. First, we introduce finite and affine W-algebras associated to truncated current Lie algebras and generalize some properties of W-algebras associated to semi-simple Lie algebras. Second, we develop an adjusted version of semi-infinite cohomology, which helps us to define affine W-algebras associated to general nilpotent elements in a uniform way. Finally, we consider vertex operator algebras in general, and show that their higher level Zhu algebras are all isomorphic to subquotients of their universal enveloping algebras.

Contents

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Notation

$\mathbb{Z}_{\geq 0}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$	Non-negative integers, integers, real numbers and complex numbers.
$\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots$	Finite-dimensional Lie algebras over $\mathbb{C}$.
$U(\mathfrak{a})$	The universal enveloping algebra of the Lie algebra $\mathfrak{a}$.
$Z(\mathfrak{a})$	The center of $U(\mathfrak{a})$.
$\mathfrak{g}, \mathfrak{g}^*$	A finite-dimensional semi-simple Lie algebra over $\mathbb{C}$ and its dual.
$(\cdot \mid \cdot)$	A non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$.
e	A non-zero nilpotent element of $\mathfrak{g}$.
$\{e, f, h\}$	An $s\ell_2$-triple such that $[e, f] = h, [h, e] = 2e, [h, f] = -2f$.
$\mathfrak{g}^x$	The centralizer of x in $\mathfrak{g}$, i.e., $\mathfrak{g}^x = \{y \in \mathfrak{g} \mid [x, y] = 0\}$.
$\mathfrak{g}_p$	The level p truncated current Lie algebra associated to $\mathfrak{g}$.
$\Gamma : \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i)$	A $\mathbb{Z}$-grading of $\mathfrak{g}$.
$\Gamma_p : \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_p(i)$	A $\mathbb{Z}$-grading of $\mathfrak{g}_p$.
$(\cdot \mid \cdot)_p$	A non-degenerate invariant symmetric bilinear form on $\mathfrak{g}_p$.
H_{χ_p}	The finite W-algebra associated to $(\mathfrak{g}_p, e)$.
$W^k(\mathfrak{g}_p, e)$	The affine W-algebra associated to $(\mathfrak{g}_p, e)$.
L, L^*	A quasi-finite $\mathbb{Z}$-graded Lie algebra and its restricted dual.
$\Lambda^{\infty/2+\bullet} L^*$	The space of semi-infinite forms on L.
$H^{\infty/2+\bullet}(L, M)$	The semi-infinite cohomology of L with coefficients in M.
$\mathrm{Zhu}(V)$ or $A_0(V)$	The Zhu algebra of a vertex algebra V.
$A_n(V)$	Higher level Zhu algebras of V.
$U(V)$	The universal enveloping algebra of V.

Tensor products are taken over $\mathbb{C}$ except other declaration.

Introduction

0.1. Given a complex finite-dimensional semi-simple Lie algebra $\mathfrak{g}$ with a non-degenerate invariant symmetric bilinear form $(\cdot \mid \cdot)$, one can construct: (1) the universal enveloping algebra $U(\mathfrak{g})$; (2) the coordinate ring $\mathbb{C}[\mathfrak{g}^*]$ of the Poisson variety $\mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual of $\mathfrak{g}$; (3) the level k vacuum representation $V^k(\mathfrak{g})$ of $\tilde{\mathfrak{g}}$, where $\tilde{\mathfrak{g}}$ is the Kac-Moody affinization of $\mathfrak{g}$; (4) the coordinate ring of the arc space $J\mathfrak{g}^*$ of $\mathfrak{g}^*$. Among them, $U(\mathfrak{g})$ is an associative algebra, $\mathbb{C}[\mathfrak{g}^*]$ is a Poisson algebra, $V^k(\mathfrak{g})$ is a vertex algebra and $\mathbb{C}[J\mathfrak{g}^*]$ is a Poisson vertex algebra. It is well known that there is a filtration on $U(\mathfrak{g})$ whose associated graded $\operatorname{gr} U(\mathfrak{g}) \cong S(\mathfrak{g}) \cong \mathbb{C}[\mathfrak{g}^*]$, so that $U(\mathfrak{g})$ can be considered as a quantization of $\mathfrak{g}^*$. Similarly, $V^k(\mathfrak{g})$ can be considered as a quantization of $J\mathfrak{g}^*$. The Zhu algebra functor sends $V^k(\mathfrak{g})$ to $U(\mathfrak{g})$ [FZ92] and $\mathbb{C}[J\mathfrak{g}^*]$ to $\mathbb{C}[\mathfrak{g}^*]$ [DSKV16]. Therefore, we have the left side of the following diagram [Ara17, DSK05].

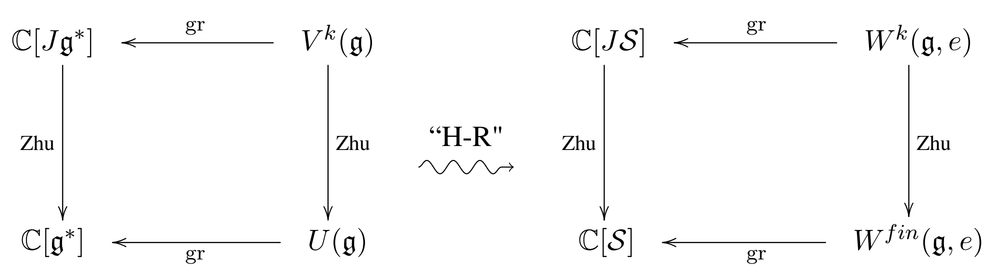


Figure 0.1: Hamiltonian reductions

Given a non-zero nilpotent element $e \in \mathfrak{g}$, one can embed it into an $s\ell_2$-triple [CM93] and perform quantum or classical Hamiltonian reductions to get the right side of the above diagram [Ara17, DS14]. The Slodowy slice $\mathcal{S}$ obtained through Poisson reduction of $\mathfrak{g}^*$ inherits a Poisson structure from $\mathfrak{g}^*$. The arc space $J\mathcal{S}$ of $\mathcal{S}$ can be considered as an infinite-dimensional Poisson variety. The coordinate rings of $\mathcal{S}$ and $J\mathcal{S}$ are called classical finite and affine W-algebras, respectively. While $\mathbb{C}[\mathcal{S}]$ is a Poisson algebra, $\mathbb{C}[J\mathcal{S}]$ is a Poisson vertex algebra [Ara12]. The quantum finite and affine W-algebras $W^{fin}(\mathfrak{g}, e)$ and $W^k(\mathfrak{g}, e)$ are quantizations of $\mathcal{S}$ [GG02, Pre02] and $J\mathcal{S}$ [DSK06], respectively. Classical and quantum finite W-algebras were proved to be the Zhu algebras of classical and quantum affine W-algebras (see [Ara07] for the principal case, and [DSK06, DSKV16] for the general case).

Convention: Whenever we refer to finite or affine W-algebras, we mean the quantum versions. If we want to refer to the classical versions, we will say classical finite and affine W-algebras.

Finite W-algebras appeared first in B. Kostant's work [Kos78], where he considered the case of a principal (i.e., regular) nilpotent element in a semi-simple Lie algebra, and proved that the resulting algebra is isomorphic to the center of the universal enveloping algebra. Then his student T. Lynch generalized the construction to even grading nilpotent elements [Lyn79]. It was A. Premet who gave the general definition of finite W-algebras associated to arbitrary nilpotent elements [Pre02].

Affine W-algebras, though more complicated, appeared a bit earlier in [BPZ84], where the authors introduced the affine W-algebra W_3 associated to $s\ell_3$ and its principal nilpotent element. The name W-algebra, came partially from the notation W_3 used there. After that, people started to study their generalizations [FF90]. It was V. Kac, S. Roan and M. Wakimoto who gave the general definition of affine W-algebras around 2003 [KRW03], where they also realized most of the important superconformal algebras through affine W-algebras.

Classical affine W-algebras associated to principal nilpotent elements were introduced by V. Drinfeld and V. Sokolov [DS84] as algebras of local functions on infinite-dimensional Poisson manifolds, where they also constructed an integrable hierarchy of bi-Hamiltonian equations associated to each principal classical affine W-algebra. These constructions of integrable hierarchies were generalized recently to classical affine W-algebras associated to general nilpotent elements in the theory of Poisson vertex algebras [DSKV13].

Explicit generators and their products of classical finite and affine W-algebras were well-studied in [MR15, DSKV16]. However, even the explicit generators of quantum finite and affine W-algebras are not clear except some special cases [Bro11, AM17, DSKV18]. On the representation theory side, there are more interesting results. Skryabin equivalence (appendix of [Pre02]) established an equivalence between the category of Whittaker modules for Lie algebras and the category of modules for finite W-algebras. Finite-dimensional irreducible modules for finite W-algebras were also classified (see [BK06] for type A and [Los11, LO14] for general cases). A highest weight theory of finite W-algebras was also studied [BGK08]. As Zhu algebras of affine W-algebras, the representation theory of finite W-algebras are closely related to that of affine W-algebras [Ara05, Ara07].

In the literature, people usually assume that $\mathfrak{g}$ is semi-simple or reductive in the constructions of W-algebras. In this thesis, we study W-algebras associated to truncated current Lie algebras.

Given a finite-dimensional semi-simple Lie algebra $\mathfrak{g}$, the current algebra associated to $\mathfrak{g}$ is the Lie algebra $\mathfrak{g}\otimes\mathbb{C}[t]$ with Lie bracket: $[a\otimes t^m, b\otimes t^n]:=[a,b]\otimes t^{m+n}$ for $a,b\in\mathfrak{g}, m,n\in\mathbb{Z}$. The level p truncated current Lie algebra $\mathfrak{g}_p$ is the quotient $\dfrac{\mathfrak{g}\otimes\mathbb{C}[t]}{\mathfrak{g}\otimes t^{p+1}\mathbb{C}[t]}$ with Lie bracket

$$[a\otimes t^i, b\otimes t^j]=[a,b]\otimes t^{i+j}, \qquad \text{where } t^{i+j}\equiv 0 \text{ when } i+j>p.$$

In the language of jet schemes [Mus01], $\mathfrak{g}_p$ is the p-th jet scheme of $\mathfrak{g}$. In the constructions of various W-algebras, a non-degenerate invariant bilinear form and a good $\mathbb{Z}$-grading (see Definition 1.2.1) play essential roles. Given a non-degenerate invariant bilinear form on $\mathfrak{g}$, one can construct a series of non-degenerate invariant bilinear forms on $\mathfrak{g}_p$ (see Lemma 2.1.3). Moreover, a good $\mathbb{Z}$-grading of

$\mathfrak{g}$ naturally induces a good $\mathbb{Z}$-grading of $\mathfrak{g}_p$ (see Lemma 2.2.1). W-algebras associated to truncated current algebras can be defined in a similar way to the semi-simple case [He].

0.2. The notion of semi-infinite cohomology is the mathematical counterpart of BRST reduction in physics. It was first introduced by B. Feigin [Fei84] around 1984 for Lie algebras. The affine W-algebra $W^k(\mathfrak{g}, e)$ associated to a principal nilpotent element can be realized as the semi-infinite cohomology of a nilpotent subalgebra of the Kac-Moody affinization of $\mathfrak{g}$ with coefficients in the vacuum representation $V^k(\mathfrak{g})$ [FKW92]. General affine W-algebras, however, differ a bit as there is an extra part in the definition of the cohomology complex [KRW03]. Computing semi-infinite cohomology requires that the Lie algebra in question admits a semi-infinite structure. Motivated by the construction of general affine W-algebras, we develop an adjusted version of semi-infinite cohomology to include the cases where the Lie algebra does not admit a semi-infinite structure but satisfies a mild condition. We also give a characterization of the differential in the (adjusted) semi-infinite cohomology [He17a]. Moreover, we show that general affine W-algebras can also be realized as semi-infinite cohomology with coefficients not in the vacuum module but in a different module.

0.3. The Zhu algebra of a vertex operator algebra was first introduced by Y. C. Zhu [FZ92]. It plays very important roles in the representation theory of vertex algebras. There is a one-to-one correspondence between the isomorphism classes of irreducible admissible modules of the vertex operator algebra and the isomorphism classes of irreducible modules of its Zhu algebra [FZ92]. Around 1998, C. Dong, H. Li and G. Mason [DLM98] generalized Zhu algebra to a series of associative algebras which we call higher level Zhu algebras. They proved similar results about the correspondence between representations of higher level Zhu algebras and those of the vertex operator algebra. Frenkel and Zhu observed that the Zhu algebra is isomorphic to a subquotient of the universal enveloping algebra [FZ92]. We show that their observation can be generalized to higher level Zhu algebras [He17b].

0.4. The organization of the thesis is as follows. In Chapter 1, we recall some preliminaries on Poisson geometry and good $\mathbb{Z}$-gradings of finite-dimensional Lie algebras. In Chapter 2, we define finite W-algebras associated to truncated current Lie algebras and show that they are quantizations of Slodowy slices. We also show that Skryabin equivalence and Kostant's theorem hold in the truncated current setting. In Chapter 3, we develop an adjusted version of semi-infinite cohomology. In Chapter 4, we define affine W-algebras associated to truncated current Lie algebras. In Chapter 5, we show that higher level Zhu algebras of a vertex operator algebra are isomorphic to subquotients of its universal enveloping algebra.

Chapter 1

Preliminaries

1.1 Poisson geometry

Let $\mathbb{K}$ be the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$.

1.1.1 Poisson manifold

Definition 1.1.1. A *Poisson algebra* over $\mathbb{K}$ is a commutative associative $\mathbb{K}$-algebra $(A, \cdot)$ with an additional bilinear binary operation $\{\cdot,\cdot\}$, which is called a *Poisson bracket*, such that $(A, \{\cdot,\cdot\})$ is a Lie algebra and the two operations $\{\cdot,\cdot\}$ and $\cdot$ satisfy Leibniz's rule:

$$\{a, b \cdot c\} = \{a, b\} \cdot c + b \cdot \{a, c\} \text{ for all } a, b, c \in A.$$

Let $(A_1, \cdot_1, \{\cdot,\cdot\}_1)$ and $(A_2, \cdot_2, \{\cdot,\cdot\}_2)$ be two Poisson algebras over $\mathbb{K}$. A *Poisson algebra homomorphism* is $\mathbb{K}$-linear map $\varphi : A_1 \to A_2$ which satisfies

$$\varphi(a \cdot_1 b) = \varphi(a) \cdot_2 \varphi(b) \text{ and } \varphi(\{a, b\}_1) = \{\varphi(a), \varphi(b)\}_2 \text{ for all } a, b \in A_1.$$

Definition 1.1.2. A real (resp. complex) *Poisson manifold* is smooth (resp. complex) manifold M, such that the ring of smooth (resp. holomorphic) functions $C^\infty(M)$ (resp. $\mathcal{O}(M)$) on M admits a Poisson algebra structure over $\mathbb{R}$ (resp. over $\mathbb{C}$).

Let M be a smooth manifold. A *smooth bi-vector field* P on M is a smooth section of the bundle $\Lambda^2 TM \to M$, where TM is the tangent bundle of M and $\Lambda^2 TM$ its second exterior power. Here a smooth section means a smooth assignment of an element $P_m \in \Lambda^2 T_m M$ to each point m of M, where $T_m M$ is the tangent space of M at m. In local coordinates $\{U, x_1, \cdots, x_n\}$, P can be expressed as

$$P_x = \sum_{i,j=1}^{n} K_{i,j}(x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \text{ for all } x \in U,$$

where $K_{i,j}(x)$ are skew-symmetric smooth functions on U, i.e., $K_{i,j}(x) = -K_{j,i}(x)$ for all $x \in U$.

Let $\Omega^1(M)$ be the space of smooth 1-forms on M, which is the collection of smooth sections of the cotangent bundle $T^*M \to M$. Let $\mathrm{Vect}(M)$ be the space of smooth vector fields on M, which is the collection of smooth sections of the tangent bundle $TM \to M$. Given a smooth bi-vector field P, we can define a map from $\Omega^1(M)$ to $\mathrm{Vect}(M)$ by the natural pairing between the cotangent space T_x^*M and the tangent space T_xM. In local coordinates $\{U, x_1, \cdots, x_n\}$, a smooth 1-form and a smooth vector field can be expressed as $\sum_i^n F_i(x)dx_i$ and $\sum_i^n G_i(x)\frac{\partial}{\partial x_i}$ respectively, where $F_i(x)$ and $G_i(x)$ are smooth functions on U. The map induced by the bi-vector field P is given by

$$P_x : \sum_i^n F_i(x)dx_i \mapsto \sum_{i,j=1}^n K_{i,j}(x)F_j(x)\frac{\partial}{\partial x_i}. \tag{1.1}$$

There is a natural differential map $\mathrm{diff} : C^\infty(M) \to \Omega^1(M)$, which sends a smooth function f to its differential $\sum_{i=1}^n \frac{\partial f}{\partial x_i}dx_i$. Composing with the map (1.1), a bi-vector field P also defines a map $P : C^\infty(M) \to \mathrm{Vect}(M)$, which in local coordinates reads

$$P_x : f \mapsto \sum_{i,j=1}^n K_{i,j}(x)\frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i}.$$

The vector field defined by $X_f(x) := P_x(f)$ is called the *Hamiltonian vector field* associated to f with respect to P.

Recall that a smooth vector field X on M defines a linear map $X : C^\infty(M) \to C^\infty(M)$, which sends a smooth function g to $X(g)$. Given a smooth bi-vector P, we have the following map

$$\begin{aligned} P : C^\infty(M) \times C^\infty(M) &\to C^\infty(M), \\ (f, g) &\mapsto \{f, g\} := X_f(g). \end{aligned} \tag{1.2}$$

Definition 1.1.3. A *Poisson structure* on a smooth manifold M is a *Poisson bi-vector field* P, i.e., a smooth bi-vector field P such that the map (1.2) gives $C^\infty(M)$ a Poisson algebra structure. A Poisson manifold is a smooth manifold with a Poisson structure.

In the above discussion, when M is a complex manifold, and if we replace P by a holomorphic section in $\Lambda^2\Theta_M$, where Θ_M is the holomorphic tangent bundle and $\Lambda^2\Theta_M$ its second exterior power, and replace $C^\infty(M)$ by $\mathcal{O}(M)$, then a complex Poisson manifold is a complex manifold with a Poisson structure, i.e., a holomorphic section in $\Lambda^2\Theta_M$ which induces a Poisson bracket on $\mathcal{O}(M)$.

Example 1.1.4. Let $\mathfrak{a}$ be a finite-dimensional Lie algebra over $\mathbb{K}$ with basis $\{x_i\}_{i=1}^n$, and $\mathfrak{a}^*$ be the dual of $\mathfrak{a}$. Regarding $\{x_i\}_{i=1}^n$ as local coordinates on $\mathfrak{a}^*$, define a bi-vector field P on $\mathfrak{a}^*$ by

$$P_\varepsilon = \sum_{i,j=1}^n \varepsilon([x_i, x_j])\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} \quad \text{for all } \varepsilon \in \mathfrak{a}^*. \tag{1.3}$$

Then (1.3) defines a Poisson structure on $\mathfrak{a}^*$. Passing to the Poisson algebra structure on $\mathbb{K}[\mathfrak{a}^*]$, which we identify with the symmetric algebra $S(\mathfrak{a})$ of $\mathfrak{a}$, the Poisson bracket is

$$\{f,g\}(\varepsilon) = \sum_{i,j} \varepsilon([x_i, x_j]) \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \quad \text{for } f, g \in S(\mathfrak{a}) \text{ and } \varepsilon \in \mathfrak{a}^*.$$

In particular, when f, g are elements of $\mathfrak{a}$, their Poisson bracket is the Lie bracket of $\mathfrak{a}$.

1.1.2 Symplectic foliation

A *smooth* 2-*form* ω on a smooth manifold M is a smooth assignment of a bilinear form $\omega_m : T_mM \times T_mM \to \mathbb{R}$ for each point $m \in M$. It is called skew-symmetric if each ω_m is skew-symmetric. It is called non-degenerate if each ω_m is non-degenerate, and is called closed if $d\omega = 0$.

Definition 1.1.5. A *symplectic manifold* is a smooth manifold M with a *symplectic form*, i.e., a closed non-degenerate and skew-symmetric smooth 2-form ω.

Remark 1.1.6. *The hypothesis of smooth should be replaced by holomorphic in the complex case.*

Let G be an algebraic group with Lie algebra $\mathfrak{g}$. Conjugation $g : G \to G$, $g \cdot x = gxg^{-1}$ induces a G-action on the tangent space of the identity, which can be identified with $\mathfrak{g}$. This action on $\mathfrak{g}$ is called the *adjoint action*. The adjoint G-action on $\mathfrak{g}$ gives rise to the transposed coadjoint action on $\mathfrak{g}^*$. We denote by Ad and Ad^* the adjoint and coadjoint actions on $\mathfrak{g}$ and $\mathfrak{g}^*$, respectively. The orbits of the coadjoint (resp. adjoint) action of G on $\mathfrak{g}^*$ (resp. on $\mathfrak{g}$) are called coadjoint (resp. adjoint) orbits.

Example 1.1.7. Symplectic structure on a coadjoint orbit. Let $\alpha \in \mathfrak{g}^*$ and $\mathbb{O}^*_\alpha = \mathrm{Ad}^*G \cdot \alpha$ be the coadjoint orbit through α. One can define a symplectic form ω on $\mathbb{O}^*_\alpha$ in the following way. For $\xi \in \mathbb{O}^*_\alpha$, let $G^\xi = \{g \in G \mid \mathrm{Ad}^*g(\xi) = \xi\}$ be the isotropy group of ξ and $\mathfrak{g}^\xi$ its Lie algebra. The tangent space $T_\xi\mathbb{O}^*_\alpha$ can be identified with $T_\xi(G/G^\xi) \cong \mathfrak{g}/\mathfrak{g}^\xi \cong \mathrm{ad}^*\mathfrak{g}(\xi)$. Let $\omega_\xi : T_\xi\mathbb{O}^*_\alpha \times T_\xi\mathbb{O}^*_\alpha \to \mathbb{C}$ be the skew-symmetric bilinear form defined by

$$\omega_\xi(\mathrm{ad}^*x(\xi), \mathrm{ad}^*y(\xi)) := \xi([x,y]).$$

As the kernel of the bilinear form $\xi([\cdot,\cdot]) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is exactly $\mathfrak{g}^\xi$, the bilinear form ω_ξ is non-degenerate on $T_\xi\mathbb{O}^*_\alpha$. Moreover, the 2-form ω defined by the assignment $\xi \mapsto w_\xi$ is closed, hence it gives a symplectic structure on $\mathbb{O}^*_\alpha$. Different proofs for the closure of ω are given in [Kir04],

Every Poisson manifold M has a *symplectic foliation* in the sense that: (1) $M = \sqcup_\alpha S_\alpha$ decomposes as a disjoint union of submanifolds, which are called symplectic leaves; (2) the Poisson structure on M restricts to a symplectic structure on each S_α. If $x \in S_\alpha$, then S_α is called the *symplectic leaf* through x. It consists the points of M which can be connected to x by piecewise Hamiltonian paths, where a Hamiltonian path is an integral curve of the Hamiltonian vector field X_f associated to a smooth function f.

Example 1.1.8. The decomposition $\mathfrak{g}^* = \sqcup_\alpha \mathbb{O}^*_\alpha$ as coadjoint orbits is the symplectic foliation of $\mathfrak{g}^*$.

Given a non-degenerate bilinear form $(\cdot \mid \cdot)$ on $\mathfrak{g}$, one can identify $\mathfrak{g}$ with $\mathfrak{g}^*$ through $(\cdot \mid \cdot)$ and hence equip $\mathfrak{g}$ itself with a Poisson structure. When the bilinear form is invariant, i.e., $([x, y] \mid z) = (x \mid [y, z])$ for all $x, y, z \in \mathfrak{g}$, the symplectic foliation of $\mathfrak{g}$ is given by the adjoint orbits. Let $\mathbb{O}$ be an adjoint orbit and $x \in \mathbb{O}$. Then the tangent space $T_x\mathbb{O}$ of $\mathbb{O}$ at x can be identified with $[\mathfrak{g}, x]$, and the symplectic form on $T_x\mathbb{O}$ becomes

$$\omega_x([a, x], [b, x]) = (x \mid [a, b]) \text{ for } a, b \in \mathfrak{g}. \tag{1.4}$$

Theorem 1.1.9 ([Vai94]). *Let M be a Poisson manifold with the symplectic foliation given by $\sqcup_\alpha S_\alpha$. Let N be a submanifold of M such that for all α,*

(1) N is transversal to S_α, i.e., $T_nN + T_nS_\alpha = T_nM$ for all $n \in N \cap S_\alpha$.

(2) For all $n \in N \cap S_\alpha$, the subspace $T_nN \cap T_nS_\alpha$ is a symplectic subspace of T_nS_α, i.e., the symplectic form on T_nS_α is non-degenerate when restricted to $T_nN \cap T_nS_\alpha$.

Then there is an induced Poisson structure on N. The symplectic foliation of N is given by $\sqcup_\alpha(N \cap S_\alpha)$ and the symplectic form on $T_n(N \cap S_\alpha)$ for all $n \in N \cap S_\alpha$ is the restriction of the symplectic form on T_nS_α.

1.1.3 Poisson reduction

Poisson reduction is a procedure of taking a subquotient of a Poisson algebra or of a Poisson manifold, such that the resulting object has a Poisson algebra or Poisson manifold structure. It allows us to construct new Poisson algebras or Poisson manifolds from old ones.

Definition 1.1.10. Let $(A, \cdot, \{\cdot, \cdot\}_A)$ be a Poisson algebra, and I an ideal of $(A, \cdot)$. Let B be a subalgebra of A/I. We say that the triple (A, I, B) is *Poisson reducible* if there is a Poisson algebra structure $\{\cdot, \cdot\}_B$ on B, such that

$$\{a, b\}_B = \overline{\{\tilde{a}, \tilde{b}\}_A} \quad \text{for all } a, b \in B,$$

where $\tilde{a}, \tilde{b} \in A$ are arbitrary representatives of a, b in A, and $\bar{x}$ is the image of $x \in A$ in A/I. The Poisson bracket $\{\cdot, \cdot\}_B$ on B is called the reduced Poisson bracket.

We have the following diagram for a Poisson reducible triple,

$$\begin{array}{ccc} & & A \\ & & \downarrow{\scriptstyle \pi} \\ B & \subset & A/I \end{array}$$

Let $(A, \cdot, \{\cdot,\cdot\}_A)$ be a Poisson algebra, and I an ideal of $(A, \cdot)$. We denote by

$$N(I) := \{a \in A \mid \{a,b\}_A \in I \text{ for all } \ b \in I\},$$

and call it the normalizer of I. One can show that $N(I)$ is a Poisson subalgebra of A, i.e., $N(I)$ is closed under both the product $\cdot$ and the bracket $\{\cdot,\cdot\}_A$. Denote by π the canonical projection $\pi : A \twoheadrightarrow A/I$. Then (A, I, B) is a Poisson reducible triple if and only if $\pi^{-1}(B)$ is a Poisson subalgebra of $N(I)$, i.e., $\pi^{-1}(B) \subseteq N(I)$ and is closed under $\{\cdot,\cdot\}_A$.

Remark 1.1.11. *There is also a Poisson manifold version of Poisson reduction [Vai94].*

1.1.4 Quantization

Definition 1.1.12. An associative algebra B is called *$\mathbb{Z}$-filtered* if there is a filtration of subspaces $\{F_iB\}_{i\in\mathbb{Z}}$, with $F_iB \subseteq F_{i+1}B$ and $B = \bigcup_{i\in\mathbb{Z}} F_iB$, such that $F_iB \cdot F_jB \subseteq F_{i+j}B$ for all $i, j \in \mathbb{Z}$. An associative algebra B is called *$\mathbb{Z}$-graded* if there is a $\mathbb{Z}$-grading $B = \bigoplus_{i\in\mathbb{Z}} B_i$ such that $B_i \cdot B_j \subseteq B_{i+j}$ for all $i, j \in \mathbb{Z}$.

Given a $\mathbb{Z}$-filtered algebra B with filtration $\{F_iB\}_{i\in\mathbb{Z}}$, one can associate it a $\mathbb{Z}$-graded algebra $\mathrm{gr}_F B$ by setting $(\mathrm{gr}_F B)_i = \dfrac{F_iB}{F_{i-1}B}$ and with multiplication defined by

$$(a + F_{i-1}B) \cdot (b + F_{j-1}B) = a \cdot b + F_{i+j-1}B \text{ for } a \in F_iB, b \in F_jB.$$

The $\mathbb{Z}$-filtered algebra B is called *almost commutative* if the associated graded algebra $\mathrm{gr}_F B$ is commutative, i.e.,

$$a \cdot b - b \cdot a \in F_{i+j-1}B \quad \text{ for all } a \in F_iB, b \in F_jB.$$

Lemma 1.1.13. *Let B be an almost commutative associative algebra with a $\mathbb{Z}$-filtration $\{F_iB\}_{i\in\mathbb{Z}}$. Then $\mathrm{gr}_F B$ is a Poisson algebra with the Poisson bracket defined by*

$$\{a + F_{i-1}B, b + F_{j-1}B\} := a \cdot b - b \cdot a + F_{i+j-2}B \quad \textit{for } \ a \in F_iB, b \in F_jB.$$

Proof. Since $\mathrm{gr}_F B$ is already commutative, we only need to prove that $(\mathrm{gr}_F B, \{\cdot,\cdot\})$ is a Lie algebra and it satisfies Leibniz's rule. For the well-definedness, let $a' = a + s, b' = b + t$ with $s \in F_{i-1}B$ and $t \in F_{j-1}B$ be other representatives of a and b, respectively, in $\mathrm{gr}_F B$. Then we have

$$\begin{aligned} a' \cdot b' - b' \cdot a' &= a \cdot b - b \cdot a + (s \cdot b - b \cdot s + a \cdot t - t \cdot a + s \cdot t - t \cdot s) \\ &\equiv a \cdot b - b \cdot a \mod F_{i+j-2}B. \end{aligned}$$

Once $\{\cdot,\cdot\}$ is well-defined, it gives $\mathrm{gr}_F B$ a Lie algebra structure since the multiplication $\cdot$ is associative. For Leibniz's rule, assume that $c \in F_k B$. Then $b \cdot c \in F_{j+k} B$, and

$$\begin{aligned}
&\{a + F_{i-1}B, b \cdot c + F_{j+k-1}B\} \\
&\qquad = a \cdot (b \cdot c) - (b \cdot c) \cdot a + F_{i+j+k-2}B \\
&\qquad = (a \cdot b) \cdot c - (b \cdot a) \cdot c + (b \cdot a) \cdot c - (b \cdot c) \cdot a + F_{i+j+k-2}B \\
&\qquad = \{a + F_{i-1}B, b + F_{j-1}B\} \cdot c + b \cdot \{a + F_{i-1}B, c + F_{k-1}B\} + F_{i+j+k-2}B.
\end{aligned}$$

□

Definition 1.1.14. Let A be a Poisson algebra and B a $\mathbb{Z}$-filtered associative algebra with a $\mathbb{Z}$-filtration $\{F_i B\}_{i \in \mathbb{Z}}$. If there is an isomorphism between $\mathrm{gr}_F B$ and A as Poisson algebras, then we say that B is a *quantization* of A.

Remark 1.1.15. *When M is a Poisson manifold, a quantization of $C^\infty(M)$ or $\mathcal{O}(M)$ is also called a quantization of M.*

Example 1.1.16. The PBW filtration on $U(\mathfrak{a})$ is given by $U(\mathfrak{a})_n = \mathrm{span}_{\mathbb{C}}\{x_1 \cdots x_m \mid m \leq n, x_i \in \mathfrak{a}\}$. It is well-known that its associated graded algebra $\mathrm{gr}\, U(\mathfrak{a}) \cong S(\mathfrak{a}) \cong \mathbb{C}[\mathfrak{a}^*]$, where $S(\mathfrak{a})$ is the symmetric algebra of $\mathfrak{a}$ and $\mathbb{C}[\mathfrak{a}^*]$ is the coordinate ring of the Poisson variety $\mathfrak{a}^*$. Indeed, the isomorphism is a Poisson algebra isomorphism. Therefore, the universal enveloping algebra $U(\mathfrak{a})$ is a quantization of the Poisson variety $\mathfrak{a}^*$.

Example 1.1.17. Let $\mathfrak{a} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{a}(i)$ be a $\mathbb{Z}$-grading of $\mathfrak{a}$, i.e., $[\mathfrak{a}(i), \mathfrak{a}(j)] \subseteq \mathfrak{a}(i+j)$ for all $i, j \in \mathbb{Z}$. For $x \in \mathfrak{a}(i)$, define its degree to be $\deg x = 2 + i$. The Kazhdan filtration on $U(\mathfrak{a})$ associated to the $\mathbb{Z}$-grading of $\mathfrak{a}$ is given by setting $K_n U(\mathfrak{a}) = \mathrm{span}_{\mathbb{C}}\{x_1 \cdots x_m \mid \sum_i \deg x_i \leq n\}$. Assume that $x \in \mathfrak{a}(i), y \in \mathfrak{a}(j)$. Then $\deg x = 2 + i, \deg y = 2 + j$ and $\deg [x, y] = 2 + i + j$ as $[x, y] \in \mathfrak{a}(i + j)$. By an induction on the number of factors in a monomial, one can show that if $u \in K_{2+i} U(\mathfrak{a}), v \in K_{2+j} U(\mathfrak{a})$, then $[u, v] \in K_{2+i+j} U(\mathfrak{a})$, i.e., $U(\mathfrak{a})$ is almost commutative with respect to the Kazhdan filtration. The Poisson algebra isomorphism $\mathrm{gr}\, U(\mathfrak{a}) \cong S(\mathfrak{a})$ preserves the $\mathbb{Z}$-gradation, while the $\mathbb{Z}$-grading on $\mathrm{gr}\, U(\mathfrak{a})$ comes from the Kazhdan filtration, and that on $S(\mathfrak{a})$ from by setting the degree of $x \in \mathfrak{a}(i)$ to be $2 + i$.

Remark 1.1.18. *Note that the PBW filtration on $U(\mathfrak{a})$ is a $\mathbb{Z}_{\geq 0}$-filtration, while the Kazhdan filtration is a $\mathbb{Z}$-filtration.*

1.2 Good $\mathbb{Z}$-grading of finite-dimensional Lie algebras

Let $\mathfrak{a}$ be a finite-dimensional Lie algebra over $\mathbb{C}$. A *$\mathbb{Z}$-grading* of $\mathfrak{a}$ is a $\mathbb{Z}$-gradation $\mathfrak{a} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{a}(i)$, such that $[\mathfrak{a}(i), \mathfrak{a}(j)] \subseteq \mathfrak{a}(i+j)$ for all $i, j \in \mathbb{Z}$. It is called *even* if $\mathfrak{a}(i) = 0$ for all odd i.

Definition 1.2.1. Let $\Gamma : \mathfrak{a} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{a}(i)$ be a $\mathbb{Z}$-grading of $\mathfrak{a}$. An element $e \in \mathfrak{a}(2)$ is called a *good element* with respect to Γ if

$$\operatorname{ad} e : \mathfrak{a}(i) \to \mathfrak{a}(i+2) \text{ is injective for } i \leq -1 \text{ and surjective for } i \geq -1.$$

A $\mathbb{Z}$-grading of $\mathfrak{a}$ is called *good* if it admits a good element.

Given a good $\mathbb{Z}$-grading Γ and a good element e, the following properties are immediate:

(1) the element e is nilpotent;

(2) the centralizer $\mathfrak{a}^e$ of e in $\mathfrak{a}$ lies in $\bigoplus_{i\geq 0} \mathfrak{a}(i)$;

(3) $\operatorname{ad} e : \mathfrak{a}(-1) \to \mathfrak{a}(1)$ is bijective.

A *standard $s\ell_2$-triple* in a Lie algebra $\mathfrak{a}$ is a triple $\{e, f, h\} \subseteq \mathfrak{a}$ with $[e, f] = h, [h, e] = 2e$ and $[h, f] = -2f$. The subalgebra spanned by a standard $s\ell_2$-triple $\{e, f, h\}$ is isomorphic to $s\ell_2$.

Many important examples of good $\mathbb{Z}$-gradings of a finite-dimensional Lie algebra $\mathfrak{a}$ come from a standard $s\ell_2$-triple $\{e, h, f\}$. It follows from the representation theory of $s\ell_2$ that the eigenspace decomposition of $\mathfrak{a}$ with respect to $\operatorname{ad} h$ is a good $\mathbb{Z}$-grading of $\mathfrak{a}$ with a good element e. Good $\mathbb{Z}$-gradings thus obtained are called *Dynkin $\mathbb{Z}$-gradings*.

Theorem 1.2.2 (Jacobson-Morozov)**.** *Let $\mathfrak{g}$ be a finite-dimensional semi-simple Lie algebra over $\mathbb{C}$ and $e \in \mathfrak{g}$ be a non-zero nilpotent element. Then e can be embedded into a standard $s\ell_2$-triple $\{e, f, h\}$ of $\mathfrak{g}$. If $h' \in [e, \mathfrak{g}]$ satisfies that $[h', e] = 2e$, then $\{e, h'\}$ can be embedded into a standard $s\ell_2$-triple $\{e, f', h'\}$ of $\mathfrak{g}$.*

The proof of Theorem 1.2.2 is not constructive but by an induction on the dimension of $\mathfrak{g}$.

Lemma 1.2.3. *Let $\Gamma : \mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i)$ be a $\mathbb{Z}$-grading of a complex semi-simple Lie algebra $\mathfrak{g}$ and $e \in \mathfrak{g}(2)$. Then there exists $h \in \mathfrak{g}(0)$ and $f \in \mathfrak{g}(-2)$, such that $\{e, h, f\}$ form a standard $s\ell_2$-triple.*

Proof. By Theorem 1.2.2, we can embed e in an $s\ell_2$-triple, say $\{e, h, f\}$. Write $h = \sum_{i\in\mathbb{Z}} h_i$ and $f = \sum_{i\in\mathbb{Z}} f_i$ with $h_i, f_i \in \mathfrak{g}(i)$. Then $[h_i, e] = \delta_{i,0} 2e$ as $[h_i, e] \in \mathfrak{g}(i+2)$ and $[h, e] = 2e$. We also have $[e, f_i] = h_{i+2}$ as $[e, f] = h$. In particular, we have $[e, f_{-2}] = h_0$. Therefore, by Theorem 1.2.2, there exists f', such that $\{e, h_0, f'\}$ form a standard $s\ell_2$-triple. Write $f' = \sum_{i\in\mathbb{Z}} f'_i$ with $f'_i \in \mathfrak{g}(i)$, then $\{e, h_0, f'_{-2}\}$ is a standard $s\ell_2$-triple that we are looking for. □

Definition 1.2.4. Given an associative algebra A (resp. a Lie algebra L), a linear map $D : A \to A$ (resp. $D : L \to L$) is called a *derivation* of A (resp. of L) if $D(ab) = D(a)b + aD(b)$ (resp. $D([a, b]) = [D(a), b] + [a, D(b)]$) for all $a, b \in A$ (resp. $\in L$). The derivations are denoted by $\operatorname{Der} A$ or $\operatorname{Der} L$.

For a Lie algebra $\mathfrak{g}$, we have a notion of inner derivation. Let $a \in \mathfrak{g}$, and $\operatorname{ad} a : \mathfrak{g} \to \mathfrak{g}$ be the map defined by $\operatorname{ad} a(x) = [a, x]$. Then the Jacobi identity in the definition of a Lie algebra implies that $\operatorname{ad} a$ is a derivation of $\mathfrak{g}$. Such derivations are called *inner derivations*. We denote by $\operatorname{Inn}\mathfrak{g}$ the collection of inner derivations of $\mathfrak{g}$. It is well-known that $\operatorname{Der}\mathfrak{g}$ has a Lie algebra structure and $\operatorname{Inn}\mathfrak{g}$ is an ideal of $\operatorname{Der}\mathfrak{g}$. When $\mathfrak{g}$ is semi-simple, we have $\operatorname{Inn}\mathfrak{g} = \operatorname{Der}\mathfrak{g}$.

Lemma 1.2.5. *Let $\mathfrak{g}$ be a semi-simple Lie algebra and $\Gamma : \mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i)$ be a $\mathbb{Z}$-grading of $\mathfrak{g}$. Then there exists an element $h_\Gamma \in \mathfrak{g}$, such that $[h_\Gamma, x] = ix$ for all $x \in \mathfrak{g}(i)$.*

Proof. It is clear that the linear operator $\delta : \mathfrak{g} \to \mathfrak{g}$ defined by $\delta(x) = ix$ for $x \in \mathfrak{g}(i)$ is a derivation of $\mathfrak{g}$. Since all derivations of a semi-simple Lie algebra are inner, there exists an element $h_\Gamma \in \mathfrak{g}$ such that $[h_\Gamma, x] = \delta(x) = ix$ for $x \in \mathfrak{g}(i)$. □

Remark 1.2.6. *A complete classification of good $\mathbb{Z}$-gradings of finite-dimensional simple Lie algebras over $\mathbb{C}$ was given in [EK05].*

1.3 Vector superspace

A vector superspace is a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. An element $a \in V$ is said to be homogeneous if $a \in V_{\bar{i}}$ for some $\bar{i} \in \mathbb{Z}_2$, and $p(a) = \bar{i}$ is called the *parity* or *degree* of a. Given a vector superspace V, its endomorphism space $\operatorname{End} V$ is naturally $\mathbb{Z}_2$-graded by setting

$$(\operatorname{End} V)_{\bar{j}} := \{f \in \operatorname{End} V \mid f(V_{\bar{i}}) \subseteq V_{\bar{i}+\bar{j}} \text{ for } \bar{i} \in \mathbb{Z}_2\}.$$

Elements of $(\operatorname{End} V)_{\bar{0}}$ are called even endomorphisms and those of $(\operatorname{End} V)_{\bar{1}}$ odd endomorphisms. A homomorphism f between two vector superspaces V and W is said to be *parity-preserving* if $f(V_{\bar{i}}) \subseteq W_{\bar{i}}$ for $\bar{i} \in \mathbb{Z}_2$.

Degree Convention: Whenever we use the notation $p(a)$, we assume that a is homogeneous.

Definition 1.3.1. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace. A bilinear form $(\cdot \mid \cdot) : V \times V \to \mathbb{C}$ is called *supersymmetric* if $(V_{\bar{0}} \mid V_{\bar{1}}) = (V_{\bar{1}} \mid V_{\bar{0}}) = 0$ and it is symmetric on $V_{\bar{0}}$ and skew-symmetric on $V_{\bar{1}}$. It is called *skew-supersymmetric* if $(V_{\bar{0}} \mid V_{\bar{1}}) = (V_{\bar{1}} \mid V_{\bar{0}}) = 0$ and it is skew-symmetric on $A_{\bar{0}}$ and symmetric on $A_{\bar{1}}$.

A product in a superspace V is called supercommutative if $a \cdot b = (-1)^{p(a)p(b)} b \cdot a$ and is called anti-supercommutative if $a \cdot b = -(-1)^{p(a)p(b)} b \cdot a$ for all $a, b \in V$.

Sign Convention: For a superspace V, when we need to change the positions of two adjacent elements a, b in a product, we usually need to add a sign $\pm(-1)^{p(a)p(b)}$.

Chapter 2

Finite W-algebras associated to truncated current Lie algebras

In this chapter, we define finite W-algebras associated to truncated current Lie algebras and study some of their properties.

2.1 Truncated current Lie algebras

Given a finite-dimensional Lie algebra $\mathfrak{a}$, the *current algebra* associated to $\mathfrak{a}$ is the Lie algebra $\mathfrak{a}\otimes\mathbb{C}[t]$ with Lie bracket defined by $[a\otimes t^m, b\otimes t^n] := [a,b]\otimes t^{m+n}$ for $a,b\in\mathfrak{a}, m,n\in\mathbb{Z}_{\geq 0}$. One can show that the subspace $\mathfrak{a}\otimes t^p\mathbb{C}[t]$ is an ideal of $\mathfrak{a}\otimes\mathbb{C}[t]$ for any nonnegative integer p.

Definition 2.1.1. The *level p truncated current Lie algebra* associated to $\mathfrak{a}$ is the quotient Lie algebra

$$\mathfrak{a}_p := \frac{\mathfrak{a}\otimes\mathbb{C}[t]}{\mathfrak{a}\otimes t^{p+1}\mathbb{C}[t]} \cong \mathfrak{a}\otimes\frac{\mathbb{C}[t]}{t^{p+1}\mathbb{C}[t]}.$$

The Lie bracket of $\mathfrak{a}_p$ is

$$[a\otimes t^i, b\otimes t^j] = [a,b]\otimes t^{i+j}, \quad \text{where } t^{i+j}\equiv 0 \text{ when } i+j>p.$$

Remark 2.1.2. *In the language of jet schemes [Mus01], $\mathfrak{a}_p$ is the p-th jet scheme of $\mathfrak{a}$. Truncated current Lie algebras are also called generalized Takiff algebras or polynomial Lie algebras.*

For convenience, we write xt^i for $x\otimes t^i$. An element of $\mathfrak{a}_p$ can be uniquely expressed as a sum $\sum_{i=0}^p x_i t^i$ with $x_i\in\mathfrak{a}$. When $q\geq p$, the canonical surjective map $\pi_{q,p}:\mathfrak{a}_q\twoheadrightarrow\mathfrak{a}_p$ sending $\mathfrak{a}\otimes t^k$ to zero for $k\geq p+1$ is a Lie algebra homomorphism. For a subspace $\mathfrak{b}\subseteq\mathfrak{a}$, we let $\mathfrak{b}_p=\mathfrak{b}\otimes\dfrac{\mathbb{C}[t]}{t^{p+1}\mathbb{C}[t]}$, which is a subspace of $\mathfrak{a}_p$. If $\mathfrak{b}$ is a subalgebra of $\mathfrak{a}$, then $\mathfrak{b}_p$ is a subalgebra of $\mathfrak{a}_p$. For a nonnegative integer $k\leq p$, we denote by $\mathfrak{a}^{(k)}=\mathfrak{a}\otimes t^k$. By $\mathfrak{a}^{(\geq 1)}$ we mean $\bigoplus_{k\geq 1}\mathfrak{a}^{(k)}$. Then $\mathfrak{a}^{(0)}\cong\mathfrak{a}$ is a subalgebra of $\mathfrak{a}_p$ and $\mathfrak{a}^{(\geq 1)}$ is an ideal of $\mathfrak{a}_p$.

Let $(\cdot \mid \cdot)$ be a symmetric bilinear form on $\mathfrak{a}$. Let $\bar{c} := (c_0, \cdots, c_p)$ with $c_i \in \mathbb{C}$. Define a symmetric bilinear form on $\mathfrak{a}_p$ by the formula

$$(x \mid y)_p := \sum_{k=0}^{p} c_k \sum_{i+j=k} (x_i \mid y_j), \tag{2.1}$$

where $x = \sum_{i=0}^{p} x_i t^i$ and $y = \sum_{i=0}^{p} y_i t^i$ with $x_i, y_i \in \mathfrak{a}$.

Lemma 2.1.3 ([Cas11])**.** *Assume that $(\cdot \mid \cdot)$ is non-degenerate and invariant on $\mathfrak{a}$. Then the bilinear form $(\cdot \mid \cdot)_p$ defined by (2.1) is invariant and symmetric. It is non-degenerate if and only if $c_p \neq 0$.*

Proof. Let $x = \sum_i x_i t^i$, $y = \sum_i y_i t^i$ and $z = \sum_i z_i t^i$ with $x_i, y_i, z_i \in \mathfrak{a}$. For the invariance, we have

$$\begin{aligned}([x, y] \mid z)_p &= \sum_{i,j,k} c_k([x_i, y_j] \mid z_{k-i-j}) \\ &= \sum_{i,j,k} c_k(x_i \mid [y_j, z_{k-i-j}]) \\ &= \sum_{i',j,k} c_k(x_{k-j-i'} \mid [y_j, z_{i'}]) \\ &= (x \mid [y, z])_p.\end{aligned}$$

If $c_p = 0$, it is clear that $\mathfrak{a}^{(p)}$ lies in the kernel of the form $(\cdot \mid \cdot)_p$, so it is degenerate. When $c_p \neq 0$, assume that $a = \sum_{i \geq i_0} a_i t^i$, with $a_{i_0} \neq 0$. By the non-degenerancy of $(\cdot \mid \cdot)$, there exists an element $b \in \mathfrak{a}$, such that $(a_{i_0} \mid b) \neq 0$. Then $(a \mid bt^{p-i_0})_p = c_p(a_{i_0} \mid b) \neq 0$, i.e., $(\cdot \mid \cdot)_p$ is non-degenerate.

□

Lemma 2.1.4. $\operatorname{Der} \dfrac{\mathbb{C}[t]}{\langle t^{p+1} \rangle} \cong \dfrac{t\mathbb{C}[t]}{\langle t^{p+1} \rangle} \dfrac{d}{dt}$.

Proof. Given a polynomial $f(t) \in t\mathbb{C}[t]/\langle t^{p+1} \rangle$, setting $g(t) \mapsto f(t)\dfrac{d}{dt}g(t)$ defines a derivation of $\mathbb{C}[t]/\langle t^{p+1} \rangle$. Conversely, let D be a derivation of $\mathbb{C}[t]/\langle t^{p+1} \rangle$. As $\mathbb{C}[t]/\langle t^{p+1} \rangle$ is generated by $\{1, t\}$ and $D(1) = 0$, D is determined by $D(t)$. Assume that $D(t) = g(t)$ for some $g(t) \in \mathbb{C}[t]/\langle t^{p+1} \rangle$. Then Leibniz's rule implies that $D(t^k) = kt^{k-1}g(t)$, i.e., $D = g(t)\dfrac{d}{dt}$. But $(p+1)t^p g(t) = D(t^{p+1}) = 0$ implies that $g(0) = 0$, so $g(t) \in t\mathbb{C}[t]/\langle t^{p+1} \rangle$ and $D \in t\mathbb{C}[t]/\langle t^{p+1} \rangle \dfrac{d}{dt}$. □

Let M be a $\mathfrak{g}$-module. A derivation from $\mathfrak{g}$ to M is a linear map $f : \mathfrak{g} \to M$ satisfying

$$f([a, b]) = a \cdot f(b) - b \cdot f(a) \text{ for all } a, b \in \mathfrak{g}.$$

The derivations from $\mathfrak{g}$ to M is denoted by $\operatorname{Der}(\mathfrak{g}, M)$. Given an element $m \in M$, define $\operatorname{ad} m(x) = x \cdot m$ for all $x \in \mathfrak{g}$. Then the Lie algebra action of $\mathfrak{g}$ on M implies that $\operatorname{ad} m \in \operatorname{Der}(\mathfrak{g}, M)$. Such derivations are called inner derivations and are denoted by $\operatorname{Inn}(\mathfrak{g}, M)$. We have $\operatorname{Der} \mathfrak{g} = \operatorname{Der}(\mathfrak{g}, \mathfrak{g})$ and $\operatorname{Inn} \mathfrak{g} = \operatorname{Inn}(\mathfrak{g}, \mathfrak{g})$, where $\mathfrak{g}$ is considered as the adjoint module of $\mathfrak{g}$.

In the language of Lie algebra cohomology (see Section 3.1), a derivation from $\mathfrak{g}$ to M is a 1-cocycle with coefficients in M and an inner derivation from $\mathfrak{g}$ to M is a 1-coboundary with coefficients in M, so $H^1(\mathfrak{g}, M) = \mathrm{Der}(\mathfrak{g}, M)/\mathrm{Inn}(\mathfrak{g}, M)$.

Lemma 2.1.5 (Whitehead)**.** *Let $\mathfrak{g}$ be finite-dimensional semi-simple Lie algebra and M a finite-dimensional non-trivial simple $\mathfrak{g}$-module. Then $H^i(\mathfrak{g}, M) = 0$ for all $i > 0$, in particular, we have* $\mathrm{Der}(\mathfrak{g}, M) = \mathrm{Inn}(\mathfrak{g}, M)$.

Let $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g})$ and $d \in \mathrm{Der}\,\dfrac{\mathbb{C}[t]}{\langle t^{p+1}\rangle}$. Consider the map $D = \varphi \otimes d : \mathfrak{g}_p \to \mathfrak{g}_p$ defined by sending $a \otimes f(t)$ to $\varphi(a) \otimes df(t)$. We have

$$D([a \otimes f(t), b \otimes g(t)]) = D([a, b] \otimes f(t)g(t)) = \varphi([a, b]) \otimes d(f(t)g(t)).$$

Since $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g})$, we have $\varphi([a, b]) = [a, \varphi(b)] = -\varphi([b, a]) = -[b, \varphi(a)] = [\varphi(a), b]$. Since $d \in \mathrm{Der}\,\dfrac{\mathbb{C}[t]}{\langle t^{p+1}\rangle}$, we have $d(f(t)g(t)) = d(f(t))g(t) + f(t)d(g(t))$. Therefore, we have

$$\begin{aligned} D([a \otimes f(t), b \otimes g(t)]) &= \varphi([a, b]) \otimes (d(f(t))g(t) + f(t)d(g(t))) \\ &= [\varphi(a), b] \otimes d(f(t))g(t) + [a, \varphi(b)] \otimes f(t)d(g(t)) \\ &= [D(a \otimes f(t)), b \otimes g(t)] + [a \otimes f(t), D(b \otimes g(t))], \end{aligned}$$

i.e., $\varphi \otimes d \in \mathrm{Der}\,\mathfrak{g}_p$.

Proposition 2.1.6. *Let $\mathfrak{g}$ be a finite-dimensional semi-simple Lie algebra. Then*

$$\mathrm{Der}\,\mathfrak{g}_p \cong \left(\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g}) \otimes \mathrm{Der}\frac{\mathbb{C}[t]}{\langle t^{p+1}\rangle}\right) \ltimes \mathrm{Inn}\,\mathfrak{g}_p.$$

Proof. Given $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g})$ and $d \in \mathrm{Der}\,\dfrac{\mathbb{C}[t]}{\langle t^{p+1}\rangle}$, we have $(\varphi \otimes d)(\mathfrak{g}^{(0)}) = 0$, so every element of $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g}) \otimes \mathrm{Der}\dfrac{\mathbb{C}[t]}{\langle t^{p+1}\rangle}$ kills $\mathfrak{g}^{(0)}$. But we have $\mathrm{ad}\,x(\mathfrak{g}^{(0)}) \neq 0$ for all $x \in \mathfrak{g}_p$ which is non-zero, so

$$\mathrm{Inn}\,\mathfrak{g}_p \cap \left(\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g}) \otimes \mathrm{Der}\frac{\mathbb{C}[t]}{\langle t^{p+1}\rangle}\right) = 0.$$

We know that $\mathrm{Inn}\,\mathfrak{g}_p$ is an ideal of $\mathrm{Der}\,\mathfrak{g}_p$, so we only need to prove that

$$\mathrm{Der}\,\mathfrak{g}_p = \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g}) \otimes \mathrm{Der}\frac{\mathbb{C}[t]}{\langle t^{p+1}\rangle} + \mathrm{Inn}\,\mathfrak{g}_p.$$

For $0 \leq i \leq p$, let π_i be the projection of $\mathfrak{g}_p$ to the subspace $\mathfrak{g}^{(i)}$, i.e., $\pi_i(\sum_{k=0}^{p} x_k t^k) = x_i t^i$.

Note that $\mathfrak{g}_p$ is generated by $\mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$, so a derivation $D \in \mathrm{Der}\,\mathfrak{g}_p$ is determined by its value on $\mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$. Let $D_i = \pi_i \circ D$. Then we have $D = \sum_{i=0}^{p} D_i$. Composing π_i with Leibniz's rule, we get

$$D_i([a \otimes 1, b \otimes 1]) = [D_i(a \otimes 1), b \otimes 1] + [a \otimes 1, D_i(b \otimes 1)].$$

That means, when restricted to $\mathfrak{g}^{(0)}$, $D_i \in \mathrm{Der}(\mathfrak{g}^{(0)}, \mathfrak{g}^{(i)})$. Since $\mathfrak{g}^{(0)} \cong \mathfrak{g}$ is semi-simple, we have $\mathrm{Der}(\mathfrak{g}^{(0)}, \mathfrak{g}^{(i)}) = \mathrm{Inn}(\mathfrak{g}^{(0)}, \mathfrak{g}^{(i)})$ by Lemma 2.1.5. Therefore, there exists $x_i \otimes t^i \in \mathfrak{g} \otimes t^i$ for each $0 \leq i \leq p$, such that $\mathrm{ad}\,(x_i \otimes t^i) = D_i$ when restricted to $\mathfrak{g}^{(0)}$. Let $D' = D - \sum_{i=0}^{p} \mathrm{ad}\,(x_i \otimes t^i)$. Then $D'|_{\mathfrak{g}^{(0)}} = 0$. Let $D'_i = \pi_i \circ D'$. Applying D' to $[a \otimes 1, b \otimes t]$ and composing with π_i, by Leibniz's rule, we get

$$D'_i([a \otimes 1, b \otimes t]) = [a \otimes 1, D'_i(b \otimes t)]. \tag{2.2}$$

When restricted to $\mathfrak{g}^{(1)}$, (2.2) implies that $D'_i : \mathfrak{g}^{(1)} \to \mathfrak{g}^{(i)}$ is a $\mathfrak{g}^{(0)}$-module homomorphism. As $\mathfrak{g}^{(1)} \cong \mathfrak{g}^{(i)} \cong \mathfrak{g}$ as $\mathfrak{g}$-modules, there exist $\mathfrak{g}$-module homomorphisms $\varphi_i : \mathfrak{g} \to \mathfrak{g}$ such that $D'_i = \varphi_i \otimes t^{i-1}$ when restricted to $\mathfrak{g}^{(1)}$. Note that for $i \geq 1$, $D'_i = \varphi_i \otimes t^i \dfrac{d}{dt} \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g}) \otimes \mathrm{Der}\dfrac{\mathbb{C}[t]}{\langle t^{p+1} \rangle}$, when D'_i is restricted to $\mathfrak{g}^{(1)}$. Let $D'' = D' - \sum_{i \geq 1}^{p} \varphi_i \otimes t^i \dfrac{d}{dt}$. Then $D''|_{\mathfrak{g}^{(0)}} = 0$ and $D''(\mathfrak{g}^{(1)}) \subseteq \mathfrak{g}^{(0)}$. We show that $D'' = 0$. Note that we have $D'' = D'_0 = \varphi_0 \otimes t^{-1}$ when restricted to $\mathfrak{g}^{(1)}$, where $\varphi_0 : \mathfrak{g}^{(1)} \to \mathfrak{g}^{(0)}$ is a $\mathfrak{g}^{(0)}$-module homomorphism. By Leibniz's rule, we have

$$\begin{aligned} D''([a \otimes t, b \otimes t]) &= [D''(a \otimes t), b \otimes t] + [a \otimes t, D''(b \otimes t)] \\ &= [\varphi_0(a), b] \otimes t + [a, \varphi_0(b)] \otimes t \\ &= \varphi_0[a, b] \otimes 2t. \end{aligned}$$

Since $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, we have $D''(a \otimes t^2) = \varphi_0(a) \otimes 2t$ for all $a \in \mathfrak{g}$. Inductively, we have $D''(a \otimes t^k) = \varphi_0(a) \otimes kt^{k-1}$. In particular, $D''(a \otimes t^{p+1}) = \varphi_0(a) \otimes pt^p$ for all $a \in \mathfrak{g}$. Since $a \otimes t^{p+1} = 0$ in $\mathfrak{g}_p$, we have $\varphi_0(a) = 0$ for all $a \in \mathfrak{g}$, i.e., $D'' = 0$, and

$$D = \sum_{i=1}^{p} \mathrm{ad}\,(x_i \otimes t^i) + \sum_{i \geq 1}^{p} \varphi_i \otimes t^i \frac{d}{dt} \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g}) \otimes \mathrm{Der}\frac{\mathbb{C}[t]}{\langle t^{p+1} \rangle} + \mathrm{Inn}\,\mathfrak{g}_p.$$

□

2.2 Finite W-algebras via Whittaker model definition

Let $\mathfrak{g}$ be a finite-dimensional semi-simple Lie algebra over $\mathbb{C}$ with a non-degenerate invariant symmetric bilinear form $(\cdot \mid \cdot)$. By Lemma 2.1.3, there exists a non-degenerate invariant symmetric bilinear form $(\cdot \mid \cdot)_p$ on $\mathfrak{g}_p$, which we fix from now on.

Let $\Gamma : \mathfrak{g} \overset{\mathrm{ad}\, h_\Gamma}{=\!=} \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ be a good $\mathbb{Z}$-grading of $\mathfrak{g}$ with a good element $e \in \mathfrak{g}(2)$, and $\{e, f, h\}$ an $s\ell_2$-triple containing e with $h \in \mathfrak{g}(0)$ and $f \in \mathfrak{g}(-2)$. Let $\mathfrak{g}_p(i) := \{x \in \mathfrak{g}_p \mid [h_\Gamma, x] = ix\}$. Then $\Gamma_p : \mathfrak{g}_p = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_p(i)$ is a $\mathbb{Z}$-grading of $\mathfrak{g}_p$.

Lemma 2.2.1. *The $\mathbb{Z}$-grading Γ_p of $\mathfrak{g}_p$ is good with good element e.*

Proof. Note that $\mathfrak{g}_p(i) = \mathfrak{g}(i)_p$. For the map $\mathrm{ad}\, e : \mathfrak{g}_p(i) \to \mathfrak{g}_p(i+2)$, we have $\ker \mathrm{ad}\, e = (\mathfrak{g}(i)^e)_p$ and $\mathrm{im}\, \mathrm{ad}\, e = ([\mathfrak{g}(i), e])_p$, so it is injective for $i \leq -1$ and surjective for $i \geq -1$ as e is a good element with respect to Γ. □

Remark 2.2.2. *We call Γ_p a good $\mathbb{Z}$-grading of $\mathfrak{g}_p$ induced from a good $\mathbb{Z}$-grading of $\mathfrak{g}$.*

Example 2.2.3. In this example, we show that not every good $\mathbb{Z}$-grading of $\mathfrak{g}_p$ is induced from a good $\mathbb{Z}$-grading of $\mathfrak{g}$ as in Lemma 2.2.1. Let $\mathfrak{g} = s\ell_2$ with canonical basis $\{e, f, h\}$ such that $[e, f] = h, [h, e] = 2e, [h, f] = -2f$. Consider $\mathfrak{g}_2$, which has a basis $\{e, f, h, e \otimes t, f \otimes t, h \otimes t\}$. Let $x = h + 2e \otimes t + 2f \otimes t$. Then with respect to $\operatorname{ad} x$, we have the $\mathbb{Z}$-grading on $\mathfrak{g}_2$

$$\mathfrak{g}_2 = \mathfrak{g}_2(-2) \oplus \mathfrak{g}_2(0) \oplus \mathfrak{g}_2(2) \tag{2.3}$$

with $\mathfrak{g}_2(-2) = \operatorname{span}_{\mathbb{C}}\{f \otimes t, f - h \otimes t\}, \mathfrak{g}_2(0) = \operatorname{span}_{\mathbb{C}}\{h \otimes t, h + 2e \otimes t + 2f \otimes t\}$, and $\mathfrak{g}_2(2) = \operatorname{span}_{\mathbb{C}}\{e \otimes t, e - h \otimes t\}$. It is easy to check that $e - h \otimes t$ is a good element with respect to (2.3).

Moreover, Jacobson-Morozov's lemma does not work in truncated current Lie algebras. Indeed, when $p \geq 1$, $x \otimes t$ is nilpotent in $\mathfrak{g}_p$ for any $x \in \mathfrak{g}$ and it cannot be embedded into any $s\ell_2$-triple.

Lemma 2.2.4. *Let $\Gamma_p : \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_p(i)$ be a $\mathbb{Z}$-grading of $\mathfrak{g}_p$ induced from a good $\mathbb{Z}$-grading of $\mathfrak{g}$. We have $(\mathfrak{g}_p(i) \mid \mathfrak{g}_p(j))_p = 0$ if $i + j \neq 0$.*

Proof. Let h_Γ be the semi-simple element defining Γ_p. Let $x \in \mathfrak{g}_p(i), y \in \mathfrak{g}_p(j)$ and $i + j \neq 0$. Then $([h_\Gamma, x] \mid y)_p = -(x \mid [h_\Gamma, y])_p$, i.e., $(i+j)(x \mid y)_p = 0$. Since $i+j \neq 0$, that implies $(x \mid y)_p = 0$. □

Let $\chi_p = (e \mid \cdot)_p \in \mathfrak{g}_p^*$. Define a skew-symmetric bilinear form on $\mathfrak{g}_p(-1)$ by

$$\langle \cdot, \cdot \rangle_p : \mathfrak{g}_p(-1) \times \mathfrak{g}_p(-1) \to \mathbb{C}, \qquad (x, y) \mapsto \langle x, y \rangle_p := \chi_p([x, y]). \tag{2.4}$$

Lemma 2.2.5. *The bilinear form on $\mathfrak{g}_p(-1)$ defined by (2.4) is non-degenerate.*

Proof. This follows from the surjectivity of $\operatorname{ad} e : \mathfrak{g}_p(-1) \to \mathfrak{g}_p(1)$, the invariance of the bilinear form $(\cdot \mid \cdot)_p$ and the pairing property $(\mathfrak{g}_p(i) \mid \mathfrak{g}_p(j))_p = 0$ if $i + j \neq 0$. □

Let $\mathfrak{l}_p$ be an isotropic subspace of $\mathfrak{g}_p(-1)$ with respect to the bilinear form (2.4), i.e., $(e \mid [\mathfrak{l}_p, \mathfrak{l}_p])_p = 0$. Let $\mathfrak{l}_p^\perp := \{x \in \mathfrak{g}_p(-1) \mid (e \mid [x, y])_p = 0 \text{ for all } y \in \mathfrak{l}_p\}$, and let

$$\mathfrak{m}_p := \bigoplus_{i \leq -2} \mathfrak{g}_p(i), \qquad \mathfrak{m}_{\mathfrak{l},p} := \mathfrak{m}_p \oplus \mathfrak{l}_p, \qquad \mathfrak{n}_{\mathfrak{l},p} := \mathfrak{m}_p \oplus \mathfrak{l}_p^\perp, \qquad \mathfrak{n}_p := \bigoplus_{i \leq -1} \mathfrak{g}_p(i). \tag{2.5}$$

Obviously, $\mathfrak{m}_p \subseteq \mathfrak{m}_{\mathfrak{l},p} \subseteq \mathfrak{n}_{\mathfrak{l},p} \subseteq \mathfrak{n}_p$ are all nilpotent subalgebras of $\mathfrak{g}_p$.

One can easily show that $(e \mid [\mathfrak{m}_{\mathfrak{l},p}, \mathfrak{n}_{\mathfrak{l},p}])_p = 0$, thanks to the property $(e \mid \mathfrak{g}_p(i))_p = 0$ for $i \leq -3$ and the definition of $\mathfrak{l}_p$ and $\mathfrak{l}_p^\perp$. In particular, $\chi_p = (e \mid \cdot)_p$ is a character of $\mathfrak{m}_{\mathfrak{l},p}$ hence defines a one-dimensional representation of $\mathfrak{m}_{\mathfrak{l},p}$, which we denote by $\mathbb{C}_{\chi_p}$. Let

$$Q_{\chi_p} := U(\mathfrak{g}_p) \otimes_{U(\mathfrak{m}_{\mathfrak{l},p})} \mathbb{C}_{\chi_p} \cong U(\mathfrak{g}_p)/I_{\chi_p},$$

where I_{χ_p} is the left ideal of $U(\mathfrak{g}_p)$ generated by $\{a - \chi_p(a) \mid a \in \mathfrak{m}_{\mathfrak{l},p}\}$. We denote by $\bar{u} := u + I_{\chi_p}$ for the image of $u \in U(\mathfrak{g}_p)$ in Q_{χ_p}.

Lemma 2.2.6. *The adjoint action of* $\mathfrak{n}_{\mathfrak{l},p}$ *on* $U(\mathfrak{g}_p)$ *leaves the subspace* I_{χ_p} *invariant.*

Proof. Let $x \in \mathfrak{n}_{\mathfrak{l},p}$ and $y = \sum_i u_i(a_i - \chi_p(a_i)) \in I_{\chi_p}$, with $u_i \in U(\mathfrak{g}_p)$ and $a_i \in \mathfrak{m}_{\mathfrak{l},p}$. Then

$$\begin{aligned}[x, y] &= \sum_i [x, u_i(a_i - \chi_p(a_i))] \\ &= \sum_i \left([x, u_i](a_i - \chi_p(a_i)) + u_i[x, a_i - \chi_p(a_i)]\right).\end{aligned}$$

Since $\chi_p([\mathfrak{n}_{\mathfrak{l},p}, \mathfrak{m}_{\mathfrak{l},p}]) = 0$, we have $[x, a_i - \chi_p(a_i)] = [x, a_i] \in I_{\chi_p}$, hence $[x, y] \in I_{\chi_p}$. $\square$

Since $\operatorname{ad} \mathfrak{n}_{\mathfrak{l},p}$ preserves I_{χ_p}, it induces a well-defined adjoint action on Q_{χ_p}, such that

$$[x, \bar{u}] = \overline{[x, u]} \text{ for } x \in \mathfrak{n}_{\mathfrak{l},p}, u \in U(\mathfrak{g}_p).$$

Let

$$H_{\chi_p} := Q_{\chi_p}^{\operatorname{ad} \mathfrak{n}_{\mathfrak{l},p}} = \{\bar{u} \in Q_{\chi_p} \mid [x, u] \in I_{\chi_p} \text{ for all } x \in \mathfrak{n}_{\mathfrak{l},p}\}.$$

Lemma 2.2.7. *There is a well-defined multiplication on* H_{χ_p} *by*

$$\bar{u} \cdot \bar{v} := \overline{uv} \textit{ for } \bar{u}, \bar{v} \in H_{\chi_p}.$$

Proof. First, we show that the multiplication $\bar{u} \cdot \bar{v}$ does not depend on the representatives. It is obvious that it does not depend on the representatives of v. For that of u, we need to show that $yv \in I_{\chi_p}$ for all $y \in I_{\chi_p}, \bar{v} \in H_{\chi_p}$. Assume that $y = \sum_i u_i(a_i - \chi_p(a_i))$ with $a_i \in \mathfrak{m}_{\mathfrak{l},p}$, then

$$yv = [y, v] + vy = \sum_i u_i[a_i - \chi_p(a_i), v] + \sum_i [u_i, v](a_i - \chi_p(a_i)) + vy. \tag{2.6}$$

By the definition of H_{χ_p}, we have $[a_i + \chi_p(a_i), v] = [a_i, v] \in I_{\chi_p}$ since $a_i \in \mathfrak{m}_{\mathfrak{l},p} \subseteq \mathfrak{n}_{\mathfrak{l},p}$, hence $yv \in I_{\chi_p}$.

Next we show that H_{χ_p} is closed under the multiplication. Let $\bar{u}_1, \bar{u}_2 \in H_{\chi_p}$, we need show that $\overline{u_1 u_2} \in H_{\chi_p}$, i.e., $[x, u_1 u_2] \in I_{\chi_p}$ for all $x \in \mathfrak{n}_{\mathfrak{l},p}$. By Leibniz's rule, we have

$$[x, u_1 u_2] = [x, u_1]u_2 + u_1[x, u_2].$$

By the definition of H_{χ_p}, we have $[x, u_1], [x, u_2] \in I_{\chi_p}$. Therefore, $[x, u_1]u_2 \in I_{\chi_p}$ by (2.6). $\square$

Once the multiplication is well-defined, H_{χ_p} inherits an associative algebra structure from $U(\mathfrak{g}_p)$.

Definition 2.2.8. The *finite W-algebra* $W^{fin}(\mathfrak{g}_p, e)$ associated to the pair $(\mathfrak{g}_p, e)$ is defined to be H_{χ_p}.

Remark 2.2.9. *When* $p = 0$*, we get the definition of the finite W-algebra associated to the semi-simple Lie algebra* $\mathfrak{g}$ *and the nilpotent element* e *given by A. Premet in [Pre02].*

When $\mathfrak{l}_p$ is a Lagrangian subspace, i.e., $\mathfrak{l}_p = \mathfrak{l}_p^\perp$ hence $\mathfrak{m}_{\mathfrak{l},p} = \mathfrak{n}_{\mathfrak{l},p}$, we can realize H_{χ_p} as the opposite endomorphism algebra $(\mathrm{End}_{U(\mathfrak{g}_p)} Q_{\chi_p})^{op}$ in the following way. As $Q_{\chi_p} = U(\mathfrak{g}_p)/I_{\chi_p}$ is a cyclic $\mathfrak{g}_p$-module, an endomorphism φ is determined by its value on the generator $\bar{1}$. Since $\bar{1}$ is killed by I_{χ_p}, $\varphi(\bar{1})$ must be annihilated by I_{χ_p}. On the other hand, given an element $\bar{y} \in Q_{\chi_p}$, which is killed by I_{χ_p}, $\bar{1} \mapsto \bar{y}$ defines an endomorphism of Q_{χ_p}. We thus have

$$\begin{aligned}(\mathrm{End}_{U(\mathfrak{g}_p)} Q_{\chi_p})^{op} &\cong \{\bar{y} \in Q_{\chi_p} \mid (a - \chi_p(a))y \in I_{\chi_p} \text{ for all } a \in \mathfrak{m}_{\mathfrak{l},p}\} \\ &= \{\bar{y} \in Q_{\chi_p} \mid [a, y] \in I_{\chi_p} \text{ for all } a \in \mathfrak{n}_{\mathfrak{l},p}\} \\ &= H_{\chi_p}.\end{aligned}$$

Remark 2.2.10. *When $p = 0$, it was proved that the finite W-algebras H_{χ_0} with respect to different good gradings Γ_0 [BG07] and different isotropic subspaces $\mathfrak{l}_0$ [GG02] are all isomorphic. For $p \geq 1$, we will show the independence of isotropic subspace $\mathfrak{l}_p$ in the sequel following [GG02].*

Remark 2.2.11. *As in the semi-simple case [BGK08], there are other definitions of finite W-algebras in the truncated current setting.*

2.3 Quantization of Slodowy slices

We keep the notation of Section 2.1 and Section 2.2.

2.3.1 Poisson structure on Slodowy slices

The non-degenerate invariant symmetric bilinear form $(\cdot \mid \cdot)_p$ on $\mathfrak{g}_p$ defines a bijection $\kappa_p : \mathfrak{g}_p \to \mathfrak{g}_p^*$ through $x \mapsto (x \mid \cdot)_p$. Let $\mathfrak{g}_p^f$ be the centralizer of f in $\mathfrak{g}_p$. Set

$$\mathcal{S}_{e_p} := e + \mathfrak{g}_p^f \quad \text{and} \quad \mathcal{S}_{\chi_p} := \chi_p + \ker \mathrm{ad}^* f = \kappa_p(\mathcal{S}_{e_p}).$$

When $p = 0$, $\mathcal{S}_e := \mathcal{S}_{e_0}$ is called the *Slodowy slice* through e [Slo80]. In the language of jet schemes [Mus01], $\mathcal{S}_{e_p}$ is the p-th jet scheme of $\mathcal{S}_e$. We also call S_{e_p} the Slodowy slice through e in $\mathfrak{g}_p$ and S_{χ_p} the Slodowy slice through χ_p in $\mathfrak{g}_p^*$.

By the representation theory of $s\ell_2$, we have $\mathfrak{g}_p = \mathfrak{g}_p^e \oplus [\mathfrak{g}_p, f] = \mathfrak{g}_p^f \oplus [\mathfrak{g}_p, e]$, which implies that $\mathrm{ad}\, e : [f, \mathfrak{g}_p] \xrightarrow{1:1} [e, \mathfrak{g}_p]$ and $\mathrm{ad}\, f : [e, \mathfrak{g}_p] \xrightarrow{1:1} [f, \mathfrak{g}_p]$ are both bijective.

Lemma 2.3.1. *Let $r \in \bigoplus_{i \leq 1} \mathfrak{g}_p(i)$. Then*

(a) $[e + r, [f, \mathfrak{g}_p]] \cap \mathfrak{g}_p^f = 0$.

(b) The map $\mathrm{ad}\,(e + r) : [f, \mathfrak{g}_p] \to [e + r, [f, \mathfrak{g}_p]]$ *is bijective.*

(c) If $a \in \mathfrak{g}_p$ is such that $[e + r, a] \in \mathfrak{g}_p^f$ and $(a \mid [e + r, \mathfrak{g}_p] \cap \mathfrak{g}_p^f)_p = 0$, then $[e + r, a] = 0$.

(d) $[e + r, [f, \mathfrak{g}_p]] \oplus \mathfrak{g}_p^f = [e + r, \mathfrak{g}_p] + \mathfrak{g}_p^f = \mathfrak{g}_p$.

Proof. Let $a = \sum_i a_i$ with $a_i \in \mathfrak{g}_p(i)$ such that $[f, a] \neq 0$. Let i_0 be such that $[f, a_{i_0}] \neq 0$ but $[f, a_i] = 0$ for all $i > i_0$. Then the i_0-th component (which belongs to $\mathfrak{g}_p(i_0)$) of $[e + r, [f, a]]$ is $[e, [f, a_{i_0}]]$ as $r \in \bigoplus_{i\leq 1} \mathfrak{g}_p(i)$ and $e \in \mathfrak{g}_p(2)$. Since $[f, a_{i_0}] \neq 0$ and $\operatorname{ad} e : [f, \mathfrak{g}_p] \to [e, \mathfrak{g}_p]$ is bijective, we have $[e, [f, a_{i_0}]] \neq 0$.

(a) Assume $a \in \mathfrak{g}_p$ satisfies that $0 \neq [e + r, [f, a]] \in \mathfrak{g}_p^f$. Then $[f, a] \neq 0$. Let i_0 be as above, then $0 \neq [e, [f, a_{i_0}]] \in \mathfrak{g}_p^f(i_0)$ i.e., $[f, [e, [f, a_{i_0}]]] = 0$. This contradicts to the bijectivity of $\operatorname{ad} f : [e, \mathfrak{g}_p] \to [f, \mathfrak{g}_p]$.

(b) We just need to show that $\operatorname{ad}(e + r)$ is injective on $[f, \mathfrak{g}_p]$. Suppose that $[e + r, [f, a]] = 0$ with $[f, a] \neq 0$. Let i_0 be as above. Then its i_0-th component $[e, [f, a_{i_0}]] \neq 0$, a contradiction.

(c) For a subspace V of $\mathfrak{g}_p$, we denote by $V^\perp$ its orthogonal complement with respect to $(\cdot \mid \cdot)_p$. Then $([e + r, \mathfrak{g}_p] \cap \mathfrak{g}_p^f)^\perp = [e + r, \mathfrak{g}_p]^\perp + (\mathfrak{g}_p^f)^\perp$. Note that $(\mathfrak{g}_p^f)^\perp = [f, \mathfrak{g}_p]$ and $[e + r, \mathfrak{g}_p]^\perp = \ker \operatorname{ad}(e + r)$ as $(\cdot \mid \cdot)_p$ is non-degenerate and invariant. Therefore, (c) is equivalent to saying that if $a = u+v$ with $u \in (\mathfrak{g}_p^f)^\perp = [f, \mathfrak{g}_p], v \in [e+r, \mathfrak{g}_p]^\perp$ and $[e+r, a] \in \mathfrak{g}_p^f$, then $[e+r, a] = 0$. Since $u \in [f, \mathfrak{g}_p]$ and $v \in \ker \operatorname{ad}(e + r)$, we have $[e + r, a] = [e + r, u] \in \mathfrak{g}_p^f \cap [e + r, [f, \mathfrak{g}_p]]$, which must be zero by (a).

(d) It is enough to prove $[e + r, [f, \mathfrak{g}_p]] \oplus \mathfrak{g}_p^f = \mathfrak{g}_p$. It is a direct sum because of (a). Let us count dimensions. We have $\dim[e + r, [f, \mathfrak{g}_p]] = \dim[f, \mathfrak{g}_p]$ by (b). Note that $\dim \mathfrak{g}_p^f = \dim \mathfrak{g}_p^e$ and $\dim[f, \mathfrak{g}_p] = \dim \mathfrak{g}_p - \dim \mathfrak{g}_p^e$ as we have $\mathfrak{g}_p = [\mathfrak{g}_p, f] \oplus \mathfrak{g}_p^e$, so $\dim \mathfrak{g}_p = \dim \mathfrak{g}_p^f + \dim[f, \mathfrak{g}_p]$, and (d) is proved.

□

Remark 2.3.2. *Lemma 2.3.1 was proved in [DSKV16] for* $r \in \bigoplus_{i\leq 0} \mathfrak{g}(i)$ *and* $\mathfrak{g}$ *semi-simple, where* $\Gamma : \mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i)$ *is a good* $\mathbb{Z}$*-grading of* $\mathfrak{g}$ *with a good element* $e \in \mathfrak{g}(2)$*. We have used the same argument to prove the truncated current version above.*

Combining Theorem 1.1.9 and Lemma 2.3.1, we have the following lemma.

Lemma 2.3.3. *The slice* S_{e_p} *has a Poisson structure.*

Proof. We show that the two conditions in Theorem 1.1.9 are satisfied for the submanifold S_{e_p} of $\mathfrak{g}_p$. Let $x = e + r \in S_{e_p} \cap \mathbb{O}_x$, where $\mathbb{O}_x$ is the adjoint orbit of $\mathfrak{g}_p$ through x. As $r \in \bigoplus_{i\leq 0} \mathfrak{g}_p(i)$, Lemma 2.3.1 applies. Note that $T_x S_{e_p} = \mathfrak{g}_p^f$ and $T_x\mathbb{O}_x = [\mathfrak{g}_p, x]$. Part (d) of Lemma 2.3.1 shows that S_{e_p} is transversal to $\mathbb{O}_x$ at x. Next we show that the restriction of the symplectic form ω_x defined by (1.4) on the subspace $T_x\mathbb{O}_x \cap T_x S_{e_p} = [\mathfrak{g}_p, x] \cap \mathfrak{g}_p^f$ is non-degenerate. Assume that there exists an element $[a, x] \in [\mathfrak{g}_p, x] \cap \mathfrak{g}_p^f$ such that $[a, x] \in \ker \omega_x|_{[\mathfrak{g}_p,x]\cap\mathfrak{g}_p^f}$, i.e.,

$$\omega_x([a, x], [b, x]) = (x \mid [a, b])_p = (a \mid [b, x])_p = 0$$

for all $[b, x] \in [\mathfrak{g}_p, x] \cap \mathfrak{g}_p^f$. Part (c) of Lemma 2.3.1 shows that $[a, x] = 0$. Therefore, ω_x is non-degenerate when restricted to $[\mathfrak{g}_p, x] \cap \mathfrak{g}_p^f$ and S_{e_p} inherits a Poisson structure from that of $\mathfrak{g}_p$. □

Corollary 2.3.4. The Slodowy slice $\mathcal{S}_{\chi_p}$ has a Poisson structure.

Definition 2.3.5. The *classical finite W-algebra* associated to $(\mathfrak{g}_p, e)$ is defined to be the Poisson algebra $\mathbb{C}[\mathcal{S}_{\chi_p}]$.

Remark 2.3.6. *Explicit formulas for the Poisson bracket of* $\mathbb{C}[\mathcal{S}_{\chi_p}]$ *were calculated in [DSKV16] for* $p = 0$.

2.3.2 An isomorphism of affine varieties

Let G_p be the adjoint group of $\mathfrak{g}_p$ and $N_{\mathfrak{l},p}$ the unipotent subgroup of G_p with Lie algebra $\mathfrak{n}_{\mathfrak{l},p}$. Let

$$\mathfrak{m}_{\mathfrak{l},p}^{\perp} := \{x \in \mathfrak{g}_p \mid (x \mid y)_p = 0 \text{ for all } y \in \mathfrak{m}_{\mathfrak{l},p}\}$$

be the orthogonal complement of $\mathfrak{m}_{\mathfrak{l},p}$ with respect to the bilinear form $(\cdot \mid \cdot)_p$. One can show that $\mathfrak{m}_{\mathfrak{l},p}^{\perp} = \left(\bigoplus_{i \leq 0} \mathfrak{g}_p(i)\right) \oplus [\mathfrak{l}_p^{\perp}, e]$. As $\mathfrak{n}_{\mathfrak{l},p}$ is nilpotent, the subgroup $N_{\mathfrak{l},p}$ is generated by $\exp(\operatorname{ad} x)$ with x running through $\mathfrak{n}_{\mathfrak{l},p}$. Restrict the adjoint action of $N_{\mathfrak{l},p}$ to $e + \mathfrak{m}_{\mathfrak{l},p}^{\perp}$. Assume that $y \in \mathfrak{m}_{\mathfrak{l},p}^{\perp}$. Note that

$$\exp(\operatorname{ad} x)(e + y) = (1 + \operatorname{ad} x + \cdots + \frac{\operatorname{ad}^n x}{n!} + \cdots)(e + y) \in e + \mathfrak{m}_{\mathfrak{l},p}^{\perp}.$$

Therefore, the image of the action map $N_{\mathfrak{l},p} \times (e + \mathfrak{m}_{\mathfrak{l},p}^{\perp})$ is contained in $e + \mathfrak{m}_{\mathfrak{l},p}^{\perp}$. Since $S_{e_p} \subseteq e + \mathfrak{m}_{\mathfrak{l},p}^{\perp}$, we can moreover restrict the adjoint action map to $N_{\mathfrak{l},p} \times S_{e_p}$. There is an $N_{\mathfrak{l},p}$-action on $N_{\mathfrak{l},p} \times S_{e_p}$ defined by $u \cdot (v, x) = (uv, x)$ for $u, v \in N_{\mathfrak{l},p}$ and $x \in S_{e_p}$. Note that

$$u \cdot (v, x) = (uv, x) = (uv) \cdot x = u \cdot (v \cdot x),$$

so the adjoint action map $N_{\mathfrak{l},p} \times S_{e_p} \to e + \mathfrak{m}_{\mathfrak{l},p}^{\perp}$ is $N_{\mathfrak{l},p}$-equivariant, where $N_{\mathfrak{l},p}$ acts on $e + \mathfrak{m}_{\mathfrak{l},p}^{\perp}$ by adjoint action.

Lemma 2.3.7. *The adjoint action map* $\beta : N_{\mathfrak{l},p} \times S_{e_p} \to e + \mathfrak{m}_{\mathfrak{l},p}^{\perp}$ *is an isomorphism of affine varieties.*

Proof. The adjoint action map is obviously a morphism of varieties, so we only need to show that it is bijective. Since $\mathfrak{g}_p$ has trivial center, we can identify $\mathfrak{g}_p$ with a subalgebra of $\operatorname{End} \mathfrak{g}_p$ through the map $\operatorname{ad} : \mathfrak{g}_p \to \operatorname{End} \mathfrak{g}_p$. Since ad is injective, we have $\mathfrak{n}_{\mathfrak{l},p} \cong \operatorname{ad} \mathfrak{n}_{\mathfrak{l},p}$. The adjoint group of $\mathfrak{g}_p$ is the subgroup of $\operatorname{Aut}(\mathfrak{g}_p)$ generated by $\exp(\operatorname{ad} u)$ with u running through $\mathfrak{g}_p$, and $N_{\mathfrak{l},p}$ is the subgroup generated by $\exp(\operatorname{ad} v)$ with v running through $\mathfrak{n}_{\mathfrak{l},p}$. As $\mathfrak{n}_{\mathfrak{l},p}$ is nilpotent, the exponential map $\exp : \operatorname{ad} \mathfrak{n}_{\mathfrak{l},p} \to N_{\mathfrak{l},p}$ is surjective, i.e., every element of $N_{\mathfrak{l},p}$ can be expressed as $\exp(\operatorname{ad} v)$ for some $v \in \mathfrak{n}_{\mathfrak{l},p}$. Now we show that given an element $e + z \in e + \mathfrak{m}_{\mathfrak{l},p}^{\perp}$, there exists a unique element $e + y \in S_{e_p}$ and a unique element $x \in \mathfrak{n}_{\mathfrak{l},p}$, such that $\exp(\operatorname{ad} x)(e + y) = e + z$. Note that

$$\mathfrak{m}_{\mathfrak{l},p}^{\perp} = \left(\bigoplus_{i \leq 0} \mathfrak{g}_p(i)\right) \oplus [\mathfrak{l}_p^{\perp}, e], \quad \mathfrak{n}_{\mathfrak{l},p} = \left(\bigoplus_{i \leq -2} \mathfrak{g}_p(i)\right) \oplus \mathfrak{l}_p^{\perp} \quad \text{and} \quad \mathfrak{g}_p^f \subseteq \bigoplus_{i \leq 0} \mathfrak{g}_p(i).$$

For an element $u \in \mathfrak{g}_p$, we write $u = \sum_i u_i$ with $u_i \in \mathfrak{g}_p(i)$. Let $x \in \mathfrak{n}_{\mathfrak{l},p}, y \in \mathfrak{g}_p^f$, and $z \in \mathfrak{m}_{\mathfrak{l},p}^\perp$. Then $x = \sum_{i \leq -1} x_i, y = \sum_{j \leq 0} y_j$ and $z = \sum_{k \leq 1} z_k$ with $x_{-1} \in \mathfrak{l}_p^\perp$ and $z_1 \in [\mathfrak{l}_p^\perp, e]$. Note that

$$\exp(\operatorname{ad} x)(e+y) = e + y + [x, e] + [x, y] + \sum_{n \geq 2} \frac{(\operatorname{ad} x)^n}{n!}(e+y).$$

The equation $\exp(\operatorname{ad} x)(e+y) = e + z$ means that

$$\sum_k z_k = \sum_j y_j + \sum_i [x_i, e] + \sum_{i,j} [x_i, y_j] + \sum_{n \geq 2} \frac{(\sum_i \operatorname{ad} x_i)^n}{n!}(e + \sum_j y_j), \tag{2.7}$$

which is equivalent to a series of equations, i.e., for $k \leq 1$,

$$\begin{aligned} z_k - y_k - [x_{k-2}, e] = \sum_{i+j=k} \operatorname{ad} x_i(y_j) + \sum_{n \geq 2} \frac{\sum_{i_1+\cdots+i_n=k-2} \operatorname{ad} x_{i_1} \cdots \operatorname{ad} x_{i_n}(e)}{n!} \\ + \sum_{n \geq 2} \frac{\sum_{i_1+\cdots+i_n+j=k} \operatorname{ad} x_{i_1} \cdots \operatorname{ad} x_{i_n}(y_j)}{n!}. \end{aligned} \tag{2.8}$$

We use a decreasing induction on k to show that given z, there is a unique solution (x, y) for (2.7). We remark that

- Given k, $\operatorname{ad} x_i, y_j$ appear on the right side of (2.8) only wehn $i > k-2$ and $j > k$. Moreover, if we have already found values for $\{x_i, y_j\}_{i \geq k_0-2, j \geq k_0}$ such that (2.8) is satisfied for all $k \geq k_0$, and if we only change the values of $\{x_i, y_j\}_{i < k_0-2, j < k_0}$, then (2.8) is still valid for $k \geq k_0$.
- We have the decomposition $\mathfrak{g}_p = \mathfrak{g}_p^f \oplus [\mathfrak{g}_p, e]$, i.e., $\mathfrak{g}_p(i) = \mathfrak{g}_p^f(i) \oplus [\mathfrak{g}_p(i-2), e]$, where $\mathfrak{g}_p^f(i) = \mathfrak{g}_p^f \cap \mathfrak{g}_p(i)$ for all i.
- $\operatorname{ad} e : \mathfrak{g}_p(i) \to \mathfrak{g}_p(i+2)$ is injective for $i \leq -1$.

When $k = 1$, (2.8) reads $[x_{-1}, e] = z_1$, which has a unique solution for x_{-1} when given z_1, as $x_{-1} \in \mathfrak{l}_p^\perp, z_1 \in [\mathfrak{l}_p^\perp, e]$ and $\operatorname{ad} e : \mathfrak{l}_p^\perp \to [\mathfrak{l}_p^\perp, e]$ is injective. For $k = k_0 \leq 0$, we assume that we have uniquely determined $\{x_i, y_j\}_{i \geq k_0-1, j \geq k_0+1}$ such that (2.8) is satisfied for $k \geq k_0 + 1$. We show that we can uniquely determine (x_{k_0-2}, y_{k_0}) (while $\{x_i, y_j\}_{i \geq k_0-1, j \geq k_0+1}$ will not change), such that (2.8) is satisfied for $k \geq k_0$. Set $k = k_0$ in (2.8), since the values of $\{x_i, y_j\}_{i \geq k_0-1, j \geq k_0+1}$ are already determined, the right side of (2.8) is determined, which is an element of $\mathfrak{g}_p(k_0)$. Denote it by w_{k_0}. Then (2.8) becomes $[x_{k_0-2}, e] = w_{k_0} + y_{k_0} - z_{k_0}$. This equation has a unique solution for (x_{k_0-2}, y_{k_0}) when z_{k_0} and w_{k_0} are given, as $\mathfrak{g}_p(k_0) = \mathfrak{g}_p^f(k_0) \oplus [\mathfrak{g}_p(k_0-2), e]$ and $\operatorname{ad} e$ is injective on $\mathfrak{g}_p(k_0-2)$. By induction, we can find a unique solution (x, y) for (2.7) when z is given. $\square$

Remark 2.3.8. *The above isomorphism of affine varieties was proved in [Kos78] when e is a principal nilpotent element, and then generalized by W. Gan and V. Ginzburg in [GG02] for Dynkin good $\mathbb{Z}$-grading. Their proof involves a $\mathbb{C}^*$-action on both varieties and then applies a general theorem in algebraic geometry. Our proof here is purely algebraic and works for all good $\mathbb{Z}$-gradings.*

Corollary 2.3.9. The coadjoint action map $\alpha : N_{\mathfrak{l},p} \times \mathcal{S}_{\chi_p} \to \chi_p + \mathfrak{m}_{\mathfrak{l},p}^{\perp,*}$ is an isomorphism of affine varieties, where $\mathfrak{m}_{\mathfrak{l},p}^{\perp,*} := \kappa_p(\mathfrak{m}_{\mathfrak{l},p}^\perp)$.

2.3.3 Quantization of Slodowy slices

Recall the Kazhdan filtration on $U(\mathfrak{g}_p)$ induced by the $\mathbb{Z}$-grading Γ_p in Example 1.1.17. Let $\{U_n(\mathfrak{g}_p)\}$ be the PBW-filtration on $U(\mathfrak{g}_p)$ and

$$U_n(\mathfrak{g}_p)(i) := \{x \in U_n(\mathfrak{g}_p) \mid [h_\Gamma, x] = ix\}.$$

Then $K_nU(\mathfrak{g}_p) = \sum_{i+2j\leq n} U_j(\mathfrak{g}_p)(i)$. The Kazhdan filtration is separated and exhaustive, i.e.,

$$\bigcap_{n\in\mathbb{Z}} K_nU(\mathfrak{g}_p) = \{0\} \quad \text{and} \quad U(\mathfrak{g}_p) = \bigcup_{n\in\mathbb{Z}} K_nU(\mathfrak{g}_p).$$

The Kazhdan filtration on $U(\mathfrak{g}_p)$ induces filtrations on I_{χ_p}, Q_{χ_p} and H_{χ_p}, which we also denote by K_n. Moreover, $\mathrm{gr}_K I_{\chi_p}$ is just the ideal of $\mathbb{C}[\mathfrak{g}_p^*]$ defining the affine subvariety $\chi_p + \mathfrak{m}_{\mathfrak{l},p}^{\perp,*}$, i.e., $\mathrm{gr}_K Q_{\chi_p} \cong \mathbb{C}[\chi_p + \mathfrak{m}_{\mathfrak{l},p}^{\perp,*}]$. Note that $K_nQ_{\chi_p} = 0$ for $n < 0$ as $\{a - \chi_p(a) \mid a \in \mathfrak{m}_{\mathfrak{l},p}\}$ contains all the negative-degree generators of $U(\mathfrak{g}_p)$ with respect to the Kazhdan filtration.

Since $H_{\chi_p} \subseteq Q_{\chi_p}$, we have a natural inclusion map

$$\nu_1 : \mathrm{gr}_K H_{\chi_p} \to \mathrm{gr}_K Q_{\chi_p}.$$

On the other hand, as $\mathcal{S}_{\chi_p} \subseteq \chi_p + \mathfrak{m}_{\mathfrak{l},p}^{\perp,*}$, we have a restriction map

$$\nu_2 : \mathbb{C}[\chi_p + \mathfrak{m}_{\mathfrak{l},p}^{\perp,*}] \to \mathbb{C}[\mathcal{S}_{\chi_p}].$$

Composing these two maps, we get a homomorphism, as $\mathrm{gr}_K Q_{\chi_p} \cong \mathbb{C}[\chi_p + \mathfrak{m}_{\mathfrak{l},p}^{\perp,*}]$,

$$\nu = \nu_2 \circ \nu_1 : \mathrm{gr}_K H_{\chi_p} \to \mathbb{C}[\mathcal{S}_{\chi_p}].$$

We are going to show that ν is an isomorphism.

The module Q_{χ_p} is a filtered $U(\mathfrak{n}_{\mathfrak{l},p})$-module, where the filtration on $U(\mathfrak{n}_{\mathfrak{l},p})$ is the Kazhdan filtration induced from that of $U(\mathfrak{g}_p)$. This filtration induces filtrations on the cohomologies $H^i(\mathfrak{n}_{\mathfrak{l},p}, Q_{\chi_p})$, and there are canonical homomorphisms

$$h_i : \mathrm{gr}_K H^i(\mathfrak{n}_{\mathfrak{l},p}, Q_{\chi_p}) \to H^i(\mathfrak{n}_{\mathfrak{l},p}, \mathrm{gr}_K Q_{\chi_p}). \tag{2.9}$$

Theorem 2.3.10. *The homomorphism $\nu : \mathrm{gr}_K H_{\chi_p} \to \mathbb{C}[\mathcal{S}_{\chi_p}]$ is an isomorphism.*

Proof. First, we show that $H^i(\mathfrak{n}_{\mathfrak{l},p}, \mathrm{gr}_K Q_{\chi_p}) = \delta_{i,0}\mathbb{C}[\mathcal{S}_{\chi_p}]$. Recall the isomorphism of affine varieties in Lemma 2.3.7, which is $N_{\mathfrak{l}_p}$-equivariant. Thus we have an $\mathfrak{n}_{\mathfrak{l},p}$-module isomorphism $\mathbb{C}[\chi_p + \mathfrak{m}_{\mathfrak{l},p}^{\perp,*}] \cong \mathbb{C}[N_{\mathfrak{l}_p}] \otimes \mathbb{C}[\mathcal{S}_{\chi_p}]$. Hence

$$H^i(\mathfrak{n}_{\mathfrak{l},p}, \mathrm{gr}_K Q_{\chi_p}) = H^i(\mathfrak{n}_{\mathfrak{l},p}, \mathbb{C}[\chi_p + \mathfrak{m}_{\mathfrak{l},p}^{\perp,*}]) = H^i(\mathfrak{n}_{\mathfrak{l},p}, \mathbb{C}[N_{\mathfrak{l}_p}]) \otimes \mathbb{C}[\mathcal{S}_{\chi_p}].$$

The cohomology $H^i(\mathfrak{n}_{\mathfrak{l},p}, \mathbb{C}[N_{\mathfrak{l}_p}])$ is equal to the algebraic de Rham cohomology of $N_{\mathfrak{l}_p}$ [CE48], which is $\mathbb{C}$ for $i = 0$ and trivial for $i > 0$ as $N_{\mathfrak{l}_p}$ is isomorphic to an affine space.

Next we show that the homomorphisms h_i in (2.9) are all isomorphisms. The standard cochain complex for computing the cohomology of $\mathfrak{n}_{\mathfrak{l},p}$ with coefficients in Q_{χ_p} is

$$0 \to Q_{\chi_p} \to \mathfrak{n}_{\mathfrak{l},p}^* \otimes Q_{\chi_p} \to \cdots \to \Lambda^n \mathfrak{n}_{\mathfrak{l},p}^* \otimes Q_{\chi_p} \to \cdots . \tag{2.10}$$

Recall that there is a grading on $\mathfrak{g}_p^*$ hence a grading on $\mathfrak{n}_{\mathfrak{l},p}^*$, which is positively graded as $\mathfrak{n}_{\mathfrak{l},p}$ is negatively graded in $\mathfrak{g}_p$. We write the gradation as $\mathfrak{n}_{\mathfrak{l},p}^* = \bigoplus_{i\geq 1} \mathfrak{n}_{\mathfrak{l},p}^*(i)$. Define a filtration of $\Lambda^n \mathfrak{n}_{\mathfrak{l},p}^* \otimes Q_{\chi_p}$ by setting $F_s(\Lambda^n \mathfrak{n}_{\mathfrak{l},p}^* \otimes Q_{\chi_p})$ to be the subspace spanned by $(x_1 \wedge \cdots \wedge x_n) \otimes v$ for all $x_i \in \mathfrak{n}_{\mathfrak{l},p}^*(n_i), v \in K_j Q_{\chi_p}$ such that $j + \sum n_i \leq s$, where K_j is the Kazhdan filtration on Q_{χ_p}. This defines a filtered complex on (2.10) whose associated graded complex gives us the standard cochain complex for computing the cohomology of $\mathfrak{n}_{\mathfrak{l},p}$ with coefficients in $\mathrm{gr}_K Q_{\chi_p}$.

Consider the spectral sequence with

$$E_0^{s,t} = \frac{F_s(\Lambda^{s+t}\mathfrak{n}_{\mathfrak{l},p}^* \otimes Q_{\chi_p})}{F_{s-1}(\Lambda^{s+t}\mathfrak{n}_{\mathfrak{l},p}^* \otimes Q_{\chi_p})}.$$

Then $E_1^{s,t} = H^{s+t}(\mathfrak{n}_{\mathfrak{l},p}, \dfrac{K_s Q_{\chi_p}}{K_{s-1} Q_{\chi_p}})$ and the spectral sequence converges to

$$E_\infty^{s,t} = \frac{F_s H^{s+t}(\mathfrak{n}_{\mathfrak{l},p}, Q_{\chi_p})}{F_{s-1} H^{s+t}(\mathfrak{n}_{\mathfrak{l},p}, Q_{\chi_p})},$$

i.e., the maps $h_i : \mathrm{gr}_K H^i(\mathfrak{n}_{\mathfrak{l},p}, Q_{\chi_p}) \to H^i(\mathfrak{n}_{\mathfrak{l},p}, \mathrm{gr}_K Q_{\chi_p})$ are isomorphisms hence

$$\mathrm{gr}_K H_{\chi_p} = \mathrm{gr}_K H^0(\mathfrak{n}_{\mathfrak{l},p}, Q_{\chi_p}) \cong H^0(\mathfrak{n}_{\mathfrak{l},p}, \mathrm{gr}_K Q_{\chi_p}) \cong \mathbb{C}[\mathcal{S}_{\chi_p}].$$

□

Remark 2.3.11. *For $p = 0$, the isomorphism in Theorem 2.3.10 was proved by A. Premet [Pre02] when $\mathfrak{l}$ is a Lagrangian subspace of $\mathfrak{g}(-1)$ and then generalized by W. Gan and V. Ginzburg [GG02] for general isotropic subspaces $\mathfrak{l}$. Our method here follows [GG02].*

Remark 2.3.12. *Theorem 2.3.10 shows that $(\mathbb{C}[\mathfrak{g}_p^*], \mathrm{gr}_K I_{\chi_p}, \mathbb{C}[\mathcal{S}_{\chi_p}])$ is a Poisson reducible triple and the Poisson structure on $\mathcal{S}_{\chi_p}$ can be considered as a Poisson reduction of $\mathfrak{g}_p^*$.*

Corollary 2.3.13. The algebra H_{χ_p} does not depend on the isotropic subspace $\mathfrak{l}_p$.

Proof. Let $\mathfrak{l}_p \subseteq \mathfrak{l}'_p$ be two isotropic subspaces of $\mathfrak{g}_p(-1)$, and H_{χ_p}, H'_{χ_p} the corresponding finite W-algebras. Then we have a natural map $\pi : H_{\chi_p} \hookrightarrow H'_{\chi_p}$ hence a natural map $\mathrm{gr}\,\pi : \mathrm{gr}_K H_{\chi_p} \hookrightarrow \mathrm{gr}_K H'_{\chi_p}$. By Theorem 2.3.10, we know that $\mathrm{gr}\,\pi$ is an isomorphism as they are both isomorphic to $\mathbb{C}[\mathcal{S}_{\chi_p}]$, so π is itself an isomorphism. □

Since H_{χ_p} does not depend on the isotropic subspace $\mathfrak{l}_p$, we choose it to be a Lagrangian subspace of $\mathfrak{g}_p(-1)$ from now on.

2.4 Kostant's theorem and Skryabin equivalence

2.4.1 Kostant's theorem

Given a finite-dimensional Lie algebra $\mathfrak{a}$ and a linear functional $\varphi \in \mathfrak{a}^*$, define

$$\mathfrak{a}^\varphi := \{x \in \mathfrak{a} \mid \varphi([x,y]) = 0 \text{ for all } y \in \mathfrak{a}\}.$$

The *index* of $\mathfrak{a}$ is defined to be $\chi(\mathfrak{a}) = \mathrm{Inf}\{\dim \mathfrak{a}^\varphi \mid \varphi \in \mathfrak{a}^*\}$. We say that $\varphi \in \mathfrak{a}^*$ is *regular* if $\dim \mathfrak{a}^\varphi = \chi(\mathfrak{a})$.

Given $x \in \mathfrak{a}$, let $\mathfrak{a}^x = \{y \in \mathfrak{a} \mid [x,y] = 0\}$ be the centralizer of x in $\mathfrak{a}$. Then x is called *regular* if its centralizer $\mathfrak{a}^x$ has minimal dimension, i.e., $\dim \mathfrak{a}^x \leq \dim \mathfrak{a}^{x'}$ for all $x' \in \mathfrak{a}$. When $\mathfrak{a}$ admits a non-degenerate invariant symmetric bilinear form which identifies $\mathfrak{a}$ and $\mathfrak{a}^*$, the regularity of an element is the same thing as the regularity of the corresponding linear function. It is well known that the subset of regular elements in $\mathfrak{a}$ is a dense open subset under the Zariski topology.

Let e be a regular nilpotent element in $\mathfrak{g}$, which we also call *principal nilpotent*. We show that the finite W-algebra H_{χ_p} associated to $(\mathfrak{g}_p, e)$ is isomorphic to $Z(\mathfrak{g}_p)$, the center of the universal enveloping algebra $U(\mathfrak{g}_p)$.

Let $S(\mathfrak{g}_p)$ be the symmetric algebra of $\mathfrak{g}_p$. It is well known that there is a canonical isomorphism of $\mathfrak{g}_p$-modules $\varphi : S(\mathfrak{g}_p) \to \mathrm{gr}U(\mathfrak{g}_p)$, where gr is the associated graded of the PBW filtration of $U(\mathfrak{g}_p)$. Let $I(\mathfrak{g}_p) := \{g \in S(\mathfrak{g}_p) \mid [x,g] = 0$ for all $x \in \mathfrak{g}_p\}$ be the $\mathfrak{g}_p$-invariants in $S(\mathfrak{g}_p)$ and $Z(\mathfrak{g}_p)$ be the center of $U(\mathfrak{g}_p)$. Then the restriction of φ to $I(\mathfrak{g}_p)$ yields an isomorphism of vector spaces

$$\varphi : I(\mathfrak{g}_p) \to \mathrm{gr}Z(\mathfrak{g}_p).$$

Recall that $S_{e_p} = e + \mathfrak{g}_p^f$ and $\mathcal{S}_{\chi_p} = \kappa_p(S_{e_p})$. Since $\mathcal{S}_{\chi_p} \subseteq \mathfrak{g}_p^*$, we have a canonical restriction $\iota_p : \mathbb{C}[\mathfrak{g}_p^*] \to \mathbb{C}[\mathcal{S}_{\chi_p}]$. Identifying $\mathbb{C}[\mathfrak{g}_p^*]$ with $S(\mathfrak{g}_p)$ and restricting ι_p to $I(\mathfrak{g}_p)$, we get a natural map from $I(\mathfrak{g}_p)$ to $\mathbb{C}[\mathcal{S}_{\chi_p}]$, which we still denote by ι_p.

Lemma 2.4.1 ([RT92, MS16])**.** *Let $\mathfrak{g}$ be a finite-dimensional semi-simple Lie algebra and $x = \sum_i x_i t^i \in \mathfrak{g}_p$ with $x_i \in \mathfrak{g}$. Let e be a regular nilpotent element of $\mathfrak{g}$. Then*

(1) x is regular in $\mathfrak{g}_p$ if and only if x_0 is regular in $\mathfrak{g}$.

(2) Every element of S_{e_p} is regular. Moreover, the adjoint orbit of every regular element intersects S_{e_p} in a unique point.

(3) The map $\iota_p : I(\mathfrak{g}_p) \to \mathbb{C}[\mathcal{S}_{\chi_p}]$ is an isomorphism of vector spaces.

Theorem 2.4.2. *Let e be a regular nilpotent element of $\mathfrak{g}$. Then the finite W-algebra H_{χ_p} associated to the pair $(\mathfrak{g}_p, e)$ is isomorphic to the center of $U(\mathfrak{g}_p)$.*

Proof. Since $Z(\mathfrak{g}_p) \subseteq U(\mathfrak{g}_p)$ is obviously invariant under the adjoint action of $\mathfrak{n}_{\mathfrak{l},p}$, we have a natural map $j_p : Z(\mathfrak{g}_p) \to H_{\chi_p}$, which preserves the Kazhdan filtrations on $Z(\mathfrak{g}_p)$ and H_{χ_p}. Passing to their associated graded, we have $\mathrm{gr}\, j_p : \mathrm{gr} Z(\mathfrak{g}_p) \to \mathrm{gr} H_{\chi_p}$, which is the isomorphism $\iota : I(\mathfrak{g}_p) \to \mathbb{C}[\mathcal{S}_{\chi_p}]$. Since the associated graded of j_p is an isomorphism, j_p itself is an isomorphism of algebras.

$$\begin{array}{ccc} Z(\mathfrak{g}_p) & \xrightarrow{j_p} & H_{\chi_p} \\ \Big\downarrow \mathrm{gr} & & \Big\downarrow \mathrm{gr} \\ I(\mathfrak{g}_p) & \xrightarrow[\cong]{\mathrm{gr}\, j_p} & \mathbb{C}[\mathcal{S}_{\chi_p}] \end{array}$$

□

Remark 2.4.3. *When $p = 0$, i.e., in semi-simple cases, Lemma 2.4.1 and Theorem 2.4.2 were proved by B. Kostant [Kos78]. T. Macedo and A. Savage [MS16] generalized Lemma 2.4.1 to truncated multicurrent Lie algebras, on which non-degenerate invariant bilinear forms exist. Therefore, all the lemmas and theorems in this section can be generalized to those algebras, i.e., finite W-algebras associated to truncated multicurrent Lie algebras can be defined and Kostant's theorem holds.*

Remark 2.4.4. *Explicit generators of $I(\mathfrak{g}_p)$ were constructed in [RT92], but corresponding generators of $Z(\mathfrak{g}_p)$ are not known in general. When $\mathfrak{g} = s\ell_n$, A. Molev [Mol97] has given a description of generators of $Z(\mathfrak{g}_p)$.*

2.4.2 Skryabin equivalence

Definition 2.4.5. A $\mathfrak{g}_p$-module M is called a *Whittaker module* if $a - \chi_p(a)$ acts locally nilpotently on M for all $a \in \mathfrak{m}_{\mathfrak{l},p}$. Given a Whittaker module M, an element $m \in M$ is called a *Whittaker vector* if $(a - \chi_p(a)) \cdot m = 0$ for all $a \in \mathfrak{m}_{\mathfrak{l},p}$. Let $\mathrm{Wh}(M)$ be the collection of the Whittaker vectors of M.

Lemma 2.4.6. *The $\mathfrak{g}_p$-module Q_{χ_p} is a Whittaker module, with* $\mathrm{Wh}(Q_{\chi_p}) = H_{\chi_p}$.

Proof. Remember that $Q_{\chi_p} = U(\mathfrak{g}_p)/I_{\chi_p}$, where I_{χ_p} is the left ideal of $U(\mathfrak{g}_p)$ generated by $\{a - \chi_p(a) \mid a \in \mathfrak{m}_{\mathfrak{l},p}\}$. Since $\mathfrak{m}_{\mathfrak{l},p}$ is negatively graded in the good grading Γ_p of $\mathfrak{g}_p$, it acts nilpotently on $\mathfrak{g}_p$ hence locally nilpotently on $U(\mathfrak{g}_p)$. Note that $\mathrm{ad}\, a = \mathrm{ad}\,(a - \chi_p(a))$ for all $a \in \mathfrak{m}_{\mathfrak{l},p}$, so $\mathrm{ad}\,(a - \chi_p(a))$ acts locally nilpotently on $U(\mathfrak{g}_p)$, and also on its quotient Q_{χ_p}, i.e., Q_{χ_p} is a Whittaker module. Since we choose $\mathfrak{l}_p$ to be a Lagrangian subspace of $\mathfrak{g}_p(-1)$, we have $\mathfrak{n}_{\mathfrak{l},p} = \mathfrak{m}_{\mathfrak{l},p}$. Then by the definition of H_{χ_p}, we have $\mathrm{Wh}(Q_{\chi_p}) = H^0(\mathfrak{m}_{\mathfrak{l},p}, Q_{\chi_p}) = H_{\chi_p}$. □

Let $\mathfrak{g}_p$-Wmod$^{\chi_p}$ be the category of finitely generated Whittaker $\mathfrak{g}_p$-modules and H_{χ_p}-Mod be the category of finitely generated left H_{χ_p}-modules.

Since $H_{\chi_p} \cong (\mathrm{End}_{\mathfrak{g}_p} Q_{\chi_p})^{op}$, Q_{χ_p} admits a right H_{χ_p}-module structure. Given $N \in H_{\chi_p}$-Mod, we have a $\mathfrak{g}_p$-module $Q_{\chi_p} \otimes_{H_{\chi_p}} N$ with $x \cdot (a \otimes n) := (x \cdot a) \otimes n$ for all $a \in Q_{\chi_p}, n \in N$.

Lemma 2.4.7. *(1) Let $M \in \mathfrak{g}_p$-Wmod$^{\chi_p}$. Then $\mathrm{Wh}(M) = 0$ implies that $M = 0$.*

(2) Let $M \in \mathfrak{g}_p$-Wmod$^{\chi_p}$. Then $\mathrm{Wh}(M)$ admits an H_{χ_p}-module structure, with $(y + I_{\chi_p}) \cdot v = y \cdot v$ for $y + I_{\chi_p} \in H_{\chi_p}, v \in M$.

(3) Let $N \in H_{\chi_p}$-Mod. Then $Q_{\chi_p} \otimes_{H_{\chi_p}} N \in \mathfrak{g}_p$-Wmod$^{\chi_p}$.

Proof. By definition, a Whittaker $\mathfrak{g}_p$-module M is locally $U(\mathfrak{m}_{\mathfrak{l},p})$-finite as $U(\mathfrak{m}_{\mathfrak{l},p})$ is generated by 1 and $\{a - \chi_p(a) \mid a \in \mathfrak{m}_{\mathfrak{l},p}\}$. Given a nonzero vector $v \in M$, we have $\dim U(\mathfrak{m}_{\mathfrak{l},p}) \cdot v < \infty$. Since $a - \chi_p(a)$ are nilpotent operators on $U(\mathfrak{m}_{\mathfrak{l},p}) \cdot v$, by Engel's theorem, we can find a nonzero common eigenvector for them, which is a Whittaker vector, so $\mathrm{Wh}(M) \neq 0$ if $M \neq 0$.

For (2), we only need to show that $y \cdot v \in \mathrm{Wh}(M)$ for all $y + I_{\chi_p} \in H_{\chi_p}$ and $v \in \mathrm{Wh}(M)$, because the module structure comes from the $U(\mathfrak{g}_p)$-module structure on M. We have

$$(a - \chi_p(a))y \cdot v = [a - \chi_p(a), y] \cdot v + y(a - \chi_p(a)) \cdot v = [a - \chi_p(a), y] \cdot v.$$

By the proof of Lemma 2.2.6, we have $[a, y] \in I_{\chi_p}$, so $(a - \chi_p(a))y \cdot v = 0$, i.e., $y \cdot v \in \mathrm{Wh}(M)$.

For (3), note that Q_{χ_p} is a Whittaker $\mathfrak{g}_p$-module, so $a - \chi_p(a)$ acts locally nilpotently on it. But the $U(\mathfrak{g}_p)$-action on the tensor product is from the left side, so $a - \chi_p(a)$ acts automatically locally nilpotently on the tensor product $Q_{\chi_p} \otimes_{H_{\chi_p}} N$ for all $a \in \mathfrak{m}_{\mathfrak{l},p}$. □

By Lemma 2.4.7, we have two functors,

$$\begin{aligned} \mathrm{Wh} :\quad & \mathfrak{g}_p\text{-Wmod}^{\chi_p} \longrightarrow H_{\chi_p}\text{-Mod}, && M \longmapsto \mathrm{Wh}(M), \\ Q_{\chi_p} \otimes_{H_{\chi_p}} - :\quad & H_{\chi_p}\text{-Mod} \longrightarrow \mathfrak{g}_p\text{-Wmod}^{\chi_p}, && N \longmapsto Q_{\chi_p} \otimes_{H_{\chi_p}} N. \end{aligned}$$

The functor $\mathrm{Wh}(-)$ is left exact and the functor $Q_{\chi_p} \otimes_{H_{\chi_p}} -$ is right exact.

Theorem 2.4.8. *The two functors $\mathrm{Wh}(-)$ and $Q_{\chi_p} \otimes_{H_{\chi_p}} -$ give an equivalence of categories between $\mathfrak{g}_p$-Wmod$^{\chi_p}$ and H_{χ_p}-Mod.*

Proof. Since H_{χ_p} does not depend on the isotropic subspace $\mathfrak{l}_p$, we choose it to be a Lagrangian subspace of $\mathfrak{g}_p(-1)$, so we have $\mathfrak{m}_{\mathfrak{l},p} = \mathfrak{n}_{\mathfrak{l},p}$. First, we show that $\mathrm{Wh}(Q_{\chi_p} \otimes_{H_{\chi_p}} N) \cong N$ for all $N \in H_{\chi_p}$-Mod. Assume that N is generated by a finite-dimensional subspace N_0. Setting $K_n N := (K_n H_{\chi_p}) N_0$ gives a filtration on N and it becomes a filtered H_{χ_p}-module. We twist the $\mathfrak{m}_{\mathfrak{l},p}$-action on $Q_{\chi_p} \otimes_{H_{\chi_p}} N$ by $-\chi_p$, i.e., we define a new action by

$$a \cdot (u \otimes v) = (a - \chi_p(a))u \otimes v = \mathrm{ad}(a - \chi_p(a))(u) \otimes v \quad \text{for } a \in \mathfrak{m}_{\mathfrak{l},p}, u \in Q_{\chi_p}, v \in N.$$

Then $\mathrm{Wh}(Q_{\chi_p} \otimes_{H_{\chi_p}} N) = H^0(\mathfrak{m}_{\mathfrak{l},p}, Q_{\chi_p} \otimes_{H_{\chi_p}} N)$ with respect to this new action. The Kazhdan filtrations on Q_{χ_p} and N induce a Kazhdan filtration on $Q_{\chi_p} \otimes_{H_{\chi_p}} N$, with

$$K_n(Q_{\chi_p} \otimes_{H_{\chi_p}} N) = \sum_{i+j=n} K_i Q_{\chi_p} \otimes_{H_{\chi_p}} K_j N.$$

Since both $K_n Q_{\chi_p} = 0$ and $K_n N = 0$ for $n < 0$ as we noted in Section 2.3.3, the filtration gives us homomorphisms for $i \geq 0$,

$$h_i : \mathrm{gr}_K H^i(\mathfrak{m}_{\mathfrak{l},p}, Q_{\chi_p} \otimes_{H_{\chi_p}} N) \to H^i(\mathfrak{m}_{\mathfrak{l},p}, \mathrm{gr}_K(Q_{\chi_p} \otimes_{H_{\chi_p}} N)). \tag{2.11}$$

Remember that $\mathrm{gr}_K Q_{\chi_p} \cong \mathbb{C}[\chi_p + \mathfrak{m}_{\mathfrak{l},p}^{\perp,*}]$ and $\mathrm{gr}_K H_{\chi_p} \cong \mathbb{C}[S_{\chi_p}] = \mathbb{C}[\chi_p + \ker \mathrm{ad}^* f]$. Since $\chi_p + \ker \mathrm{ad}^* f$ is an affine subspace of $\chi_p + \mathfrak{m}_{\mathfrak{l},p}^{\perp,*}$, $\mathrm{gr}_K Q_{\chi_p}$ is free over $\mathrm{gr}_K H_{\chi_p}$, and we have an isomorphism

$$\mathrm{gr}_K(Q_{\chi_p} \otimes_{H_{\chi_p}} N) \cong \mathrm{gr}_K Q_{\chi_p} \otimes_{\mathrm{gr}_K H_{\chi_p}} \mathrm{gr}_K N.$$

By Corollary 2.3.9, we have $\mathfrak{m}_{\mathfrak{l},p}$-module (precisely, $\mathfrak{n}_{\mathfrak{l},p}$-module) isomorphisms

$$\mathrm{gr}_K Q_{\chi_p} \cong \mathbb{C}[N_{\mathfrak{l}_p}] \otimes \mathbb{C}[S_{\chi_p}] \cong \mathbb{C}[N_{\mathfrak{l}_p}] \otimes \mathrm{gr}_K H_{\chi_p}.$$

Therefore,

$$\begin{aligned} H^i(\mathfrak{m}_{\mathfrak{l},p}, \mathrm{gr}_K(Q_{\chi_p} \otimes_{H_{\chi_p}} N)) &\cong H^i(\mathfrak{m}_{\mathfrak{l},p}, \mathrm{gr}_K Q_{\chi_p} \otimes_{\mathrm{gr}_K H_{\chi_p}} \mathrm{gr}_K N) \\ &\cong H^i(\mathfrak{m}_{\mathfrak{l},p}, \mathbb{C}[N_{\mathfrak{l}_p}] \otimes \mathrm{gr}_K N) \\ &\cong H^i(\mathfrak{m}_{\mathfrak{l},p}, \mathbb{C}[N_{\mathfrak{l}_p}]) \otimes \mathrm{gr}_K N \\ &= \delta_{i,0} \mathrm{gr}_K N. \end{aligned}$$

There is a spectral sequence as that in the proof of Theorem 2.3.10, which asserts that those h_i in (2.11) are all isomorphisms. Therefore, we have (note that $\mathrm{gr}_K N = N$)

$$H^i(\mathfrak{m}_{\mathfrak{l},p}, Q_{\chi_p} \otimes_{H_{\chi_p}} N) \cong \begin{cases} N & \text{for } i = 0, \\ 0 & \text{for } i \geq 1. \end{cases} \tag{2.12}$$

In particular, we have $\mathrm{Wh}(Q_{\chi_p} \otimes_{H_{\chi_p}} N) = H^0(\mathfrak{m}_{\mathfrak{l},p}, Q_{\chi_p} \otimes_{H_{\chi_p}} N) \cong N$.

Next we show that $Q_{\chi_p} \otimes_{H_{\chi_p}} \mathrm{Wh}(M) \cong M$ for all $M \in \mathfrak{g}_p\text{-}\mathrm{Wmod}^{\chi_p}$. Define a map

$$\varphi : Q_{\chi_p} \otimes_{H_{\chi_p}} \mathrm{Wh}(M) \to M, \qquad (y + I_{\chi_p}) \otimes v \mapsto y \cdot v.$$

One can show that φ is a $\mathfrak{g}_p$-module homomorphism. Then we have the following exact sequence,

$$0 \to \ker \varphi \to Q_{\chi_p} \otimes_{H_{\chi_p}} \mathrm{Wh}(M) \to M \to \mathrm{coker}\, \varphi \to 0. \tag{2.13}$$

Applying $\mathrm{Wh}(-)$ to the sequence (2.13), the identity $\mathrm{Wh}(Q_{\chi_p} \otimes_{H_{\chi_p}} \mathrm{Wh}(M)) = \mathrm{Wh}(M)$ and the left exactness of $\mathrm{Wh}(-)$ imply that $\mathrm{Wh}(\ker \varphi) = 0$, hence $\ker \varphi = 0$ by Lemma 2.4.7. Considering the long exact sequence of the cohomology of $\mathfrak{m}_{\mathfrak{l},p}$ associated to the sequence (2.13), we get

$$0 \to H^0(\mathfrak{m}_{\mathfrak{l},p}, Q_{\chi_p} \otimes_{H_{\chi_p}} \mathrm{Wh}(M)) \to H^0(\mathfrak{m}_{\mathfrak{l},p}, M) \to H^0(\mathfrak{m}_{\mathfrak{l},p}, \mathrm{coker}\, \varphi) \to 0. \tag{2.14}$$

We stop at $H^0(\mathfrak{m}_{\mathfrak{l},p}, \mathrm{coker}\varphi)$ because the next term $H^1(\mathfrak{m}_{\mathfrak{l},p}, Q_{\chi_p} \otimes_{H_{\chi_p}} \mathrm{Wh}(M)) = 0$ by (2.12). Note that $H^0(\mathfrak{m}_{\mathfrak{l},p}, -) = \mathrm{Wh}(-)$ and we already have $\mathrm{Wh}(Q_{\chi_p} \otimes_{H_{\chi_p}} \mathrm{Wh}(M)) = \mathrm{Wh}(M)$, so (2.14) implies that $\mathrm{Wh}(\mathrm{coker}\, \varphi) = 0$ hence $\mathrm{coker}\, \varphi = 0$, i.e., the map φ is an isomorphism. □

Remark 2.4.9. *Skryabin's original proof (see Appendix of [Pre02]) for Theorem 2.4.8 in the semi-simple case is different from our argument, which follows [GG02] and [Wan11].*

Chapter 3

Semi-infinite cohomology

In this chapter, we develop an adjusted version of semi-infinite cohomology which will be used to define affine W-algebras in Chapter 4. The main results of this chapter are contained in [He17a].

3.1 A brief review of Lie algebra cohomology

Let L be a complex Lie algebra and M be an L-module. The space of n-cochains (or n-forms) with coefficients in M is the space $C^n(L, M) := \mathrm{Hom}_{\mathbb{C}}(\Lambda^n L, M)$, where $\Lambda^n L$ is the n-th exterior power of L. Given an n-cochain $f \in \mathrm{Hom}_{\mathbb{C}}(\Lambda^n L, M)$, the coboundary of f is the $(n+1)$-cochain δf, defined to be

$$\begin{aligned}(\delta f)(x_1, \cdots, x_{n+1}) =& \sum_{i=1}^{n+1} (-1)^i x_i \cdot f(x_1, \cdots, \hat{x}_i, \cdots, x_{n+1}) \\ &+ \sum_{1 \le i < j \le n+1} (-1)^{i+j} f([x_i, x_j], x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_{n+1}), \end{aligned} \tag{3.1}$$

where $\hat{x}_i$ means that the term x_i is omitted and $\cdot$ is the Lie algebra action on M. One can show by straightforward calculations that $\delta^2 = 0$, hence we have a complex $(C^\bullet(L, M), \delta)$.

Definition 3.1.1. The complex $(C^\bullet(L, M), \delta)$ is called the *Chevalley-Eilenberg cochain complex* and its cohomology is called the *cohomology of L with coefficients in M*.

Let $L^* = \mathrm{Hom}_{\mathbb{C}}(L, \mathbb{C})$ be the dual of L. Assume that L is finite-dimensional, while $\{e_1, \cdots, e_d\}$ and $\{e_1^*, \cdots, e_d^*\}$ are well-ordered dual bases of L and L^*, respectively, in the sense that $\langle e_i^*, e_j \rangle = \delta_{i,j}$. One can identify $\mathrm{Hom}_{\mathbb{C}}(\Lambda^n L, M)$ with $\Lambda^n L^* \otimes M$ by considering $e_{i_1}^* \wedge \cdots \wedge e_{i_n}^* \otimes m$ as the n-cochain sending $e_{j_1} \wedge \cdots \wedge e_{j_n}$ to $\det(\langle e_{i_k}^*, e_{j_\ell} \rangle)_{1 \le k, \ell \le n} m$. If we assume that in the above expressions we have $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_n$, then

$$(e_{i_1}^* \wedge \cdots \wedge e_{i_n}^* \otimes m)(e_{j_1} \wedge \cdots \wedge e_{j_n}) = \begin{cases} m & \text{if } i_1 = j_1, \cdots, i_n = j_n, \\ 0 & \text{otherwise.} \end{cases}$$

The Clifford algebra $Cl(L \oplus L^*)$ is the associative algebra generated by $\{\iota(e_i), \varepsilon(e_i^*)\}_{1\le i\le d}$, with relations:

$$\iota(e_i)\iota(e_j) + \iota(e_j)\iota(e_i) = \varepsilon(e_j^*)\varepsilon(e_i^*) + \varepsilon(e_i^*)\varepsilon(e_j^*) = 0 \text{ and } \iota(e_i)\varepsilon(e_j^*) + \varepsilon(e_j^*)\iota(e_i) = \delta_{i,j}. \quad (3.2)$$

The Clifford algebra $Cl(L \oplus L^*)$ acts on $\Lambda^\bullet L^* = \bigoplus_{i\ge 0} \Lambda^i L^*$ in the following way: $\iota(e_i)$ is the contraction operator $\iota(e_i) : \Lambda^n L^* \to \Lambda^{n-1} L^*$ defined by

$$\iota(e_i) \cdot y_1^* \wedge \cdots \wedge y_n^* = \sum_k (-1)^{k+1} \langle y_k^*, e_i \rangle y_1^* \wedge \cdots \wedge \hat{y}_k^* \wedge \cdots \wedge y_n^*,$$

and $\varepsilon(e_i^*)$ is the wedging operator $\varepsilon(e_i^*) : \Lambda^n L^* \to \Lambda^{n+1} L^*$ defined by

$$\varepsilon(e_i^*) \cdot y_1^* \wedge \cdots \wedge y_n^* = e_i^* \wedge y_1^* \wedge \cdots \wedge y_n^*.$$

Straightforward calculations show that these operators $\iota(e_i)$ and $\varepsilon(e_i^*)$ satisfy (3.2), so it defines an action of $Cl(L \oplus L^*)$ on $\Lambda^\bullet L^*$.

Let

$$\bar{\delta} = \sum_i \varepsilon(e_i^*) \otimes e_i - \sum_{i<j} \varepsilon(e_i^*)\varepsilon(e_j^*)\iota([e_i, e_j]) \otimes 1. \quad (3.3)$$

Then $\bar{\delta} \in Cl(L \oplus L^*) \otimes U(L)$, hence it has a well-defined action on $\Lambda^\bullet L^* \otimes M$.

Proposition 3.1.2. *The operator $\bar{\delta}$ defined by (3.3) realizes the operator δ defined by (3.1) in the Chevalley-Eilenberg complex.*

Proof. We need to show that $\bar{\delta} f = \delta f$ for all $f \in \Lambda^\bullet L^* \otimes M$. It is clear that both $\bar{\delta}$ and δ map $\Lambda^n L^* \otimes M$ to $\Lambda^{n+1} L^* \otimes M$. Thus we only need to prove that for $f = e_{i_1}^* \wedge \cdots \wedge e_{i_n}^* \otimes m \in \Lambda^n L^* \otimes M$ and $\omega = e_{j_1} \wedge \cdots \wedge e_{j_{n+1}} \in \Lambda^{n+1} L$, we have $(\bar{\delta} f)(\omega) = (\delta f)(\omega)$. We assume that $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_{n+1}$. By definition,

$$\begin{aligned}(\delta f)(\omega) = &\sum_{\ell=1}^{n+1} (-1)^\ell e_{j_\ell} \cdot f(e_{j_1}, \cdots, \hat{e}_{j_\ell}, \cdots, e_{j_{n+1}}) \\ &+ \sum_{1\le k<\ell\le n+1} (-1)^{k+\ell} f([e_{j_k}, e_{j_\ell}], e_{j_1}, \cdots, \hat{e}_{j_k}, \cdots, \hat{e}_{j_\ell}, \cdots, e_{j_{n+1}}).\end{aligned}$$

Note that

$$\sum_k \varepsilon(e_k^*) \otimes e_k \cdot f = \sum_k e_k^* \wedge e_{i_1}^* \wedge \cdots \wedge e_{i_n}^* \otimes e_k \cdot m,$$

and

$$\begin{aligned}&(e_k^* \wedge e_{i_1}^* \wedge \cdots \wedge e_{i_n}^* \otimes e_k \cdot m)(\omega) \\ &\qquad = \begin{cases} (-1)^\ell e_{j_\ell} \cdot f(e_{j_1}, \cdots, \hat{e}_{j_\ell}, \cdots, e_{j_{n+1}}) & \text{if } k = j_\ell, \\ 0 & \text{if } k \notin \{j_1, \cdots, j_{n+1}\}, \end{cases}\end{aligned}$$

so

$$\left(\sum_k \varepsilon(e_k^*)\otimes e_k\cdot f\right)(\omega)=\sum_{\ell=1}^{n+1}(-1)^{\ell}e_{j_\ell}\cdot f(e_{j_1},\cdots,\hat{e}_{j_\ell},\cdots,e_{j_{n+1}}).$$

Let $f_{\hat{i}_s}=e_{i_1}^*\wedge\cdots\wedge\hat{e}_{i_s}^*\cdots\wedge e_{i_n}^*\otimes m$ and $\omega_{\hat{j}_k,\hat{j}_\ell}=e_{j_1}\wedge\cdots\wedge\hat{e}_{j_k}\cdots\wedge\hat{e}_{j_\ell}\cdots\wedge e_{j_{n+1}}$. Then

$$\varepsilon(e_i^*)\varepsilon(e_j^*)\iota([e_i,e_j])\otimes 1\cdot f=\sum_{1\le s\le n}(-1)^{s+1}\langle e_{i_s}^*,[e_i,e_j]\rangle e_i^*\wedge e_j^*\wedge f_{\hat{i}_s},$$

and

$$(e_i^*\wedge e_j^*\wedge f_{\hat{i}_s})(\omega)=\begin{cases}(-1)^{k+\ell+1}f_{\hat{i}_s}(\omega_{\hat{j}_k,\hat{j}_\ell}) & \text{if } i=j_k, j=j_\ell,\\ 0 & \text{if } \{i,j\}\nsubseteq\{j_1,\cdots,j_{n+1}\},\end{cases}$$

so

$$\begin{aligned}&\left(\sum_{i<j}\varepsilon(e_i^*)\varepsilon(e_j^*)\iota([e_i,e_j])\otimes 1\cdot f\right)(\omega)\\ &\qquad=\sum_{k<\ell}\sum_{1\le s\le n}(-1)^{s+1}(-1)^{k+\ell+1}\langle e_{i_s}^*,[e_{j_k},e_{j_\ell}]\rangle f_{\hat{i}_s}(\omega_{\hat{j}_k,\hat{j}_\ell})\\ &\qquad=\sum_{k<\ell}(-1)^{k+\ell+1}f([e_{j_k},e_{j_\ell}]\wedge\omega_{\hat{j}_k,\hat{j}_\ell})\\ &\qquad=\sum_{k<\ell}(-1)^{k+\ell+1}f([e_{j_k},e_{j_\ell}],e_{j_1},\cdots,\hat{e}_{j_k},\cdots,\hat{e}_{j_\ell},\cdots,e_{j_{n+1}}).\end{aligned}$$

Now it is clear that $(\bar{\delta}f)(\omega)=(\delta f)(\omega)$. □

3.2 Semi-infinite structure and semi-infinite cohomology

A Lie (super)algebra L is called *quasi-finite* $\mathbb{Z}$*-graded* if

$$L=\bigoplus_{n\in\mathbb{Z}}L_n \text{ with } \dim L_n<\infty, \text{ and } [L_n,L_m]\subseteq L_{m+n} \text{ for all } m,n\in\mathbb{Z}.$$

Let

$$L_{\le 0}:=\bigoplus_{n\le 0}L_n \text{ and } L_+:=\bigoplus_{n>0}L_n.$$

The $\mathbb{Z}$-grading on L induces a $\mathbb{Z}_{\le 0}$-grading on $U(L_{\le 0})$, a $\mathbb{Z}_{\ge 0}$-grading on $U(L_+)$ and a $\mathbb{Z}$-grading on $U(L)$, where $U(\mathfrak{a})$ is the universal enveloping algebra of the Lie (super)algebra $\mathfrak{a}$. By the PBW theorem, as $L=L_{\le 0}\oplus L_+$, their universal enveloping algebras, as vector spaces, are related by

$$U(L)\cong U(L_{\le 0})\otimes U(L_+).$$

A typical homogeneous element of $U(L)$ is of the form $\sum_{i=1}^r u_iv_i$ with $u_i\in U(L_{\le 0})$, $v_i\in U(L_+)$ and $\deg(u_iv_i)=\deg(u_jv_j)$ for all i,j.

Definition 3.2.1. Let L be a quasi-finite $\mathbb{Z}$-graded Lie (super)algebra. The *completion* $U(L)^{com}$ of $U(L)$ is the vector space spanned by infinite sums $\sum_{i=-\infty}^{\infty} u_i v_i$ with $u_i \in U(L_{\leq 0}), v_i \in U(L_+)$ such that only a finite number of v_i have degree less than N, i.e., $\sharp\{v_i \mid \deg v_i < N\} < \infty$, for each integer $N \in \mathbb{Z}_{\geq 0}$.

Products are well-defined in the completion, which makes $U(L)^{com}$ into an associative algebra. Obviously, $U(L)$ can be considered as a subalgebra of $U(L)^{com}$.

Definition 3.2.2. Let L be a quasi-finite $\mathbb{Z}$-graded Lie algebra. An L-module M is called *smooth* if for any given $m \in M$, we have $L_n \cdot m = 0$ for $n \gg 0$.

Remark 3.2.3. *One can extend the action of $U(L)$ on a smooth L-module to its completion $U(L)^{com}$. Let M_1, M_2 be smooth modules for L_1, L_2, respectively. Then the tensor product $M_1 \otimes M_2$ is naturally a smooth $L_1 \oplus L_2$-module.*

Definition 3.2.4. Let L_1, L_2 be two associative or Lie superalgebras, and $\varphi : L_1 \to L_2$ be an algebra homomorphism. A *superderivation* of parity $i \in \mathbb{Z}_2$ with respect to φ is a parity-preserving linear map $D : L_1 \to L_2$ satisfying Leibniz's rule

$$D(u \circ_1 v) = D(u) \circ_2 \varphi(v) + (-1)^{i \cdot p(u)} \varphi(u) \circ_2 D(v) \tag{3.4}$$

for all $u, v \in L_1$ with u homogeneous, where $p(u)$ is the parity of u and $\circ_1, \circ_2$ are the multiplications or Lie brackets of L_1, L_2, respectively. We call D even if $i = 0$ and odd if $i = 1$. When one of $\{L_1, L_2\}$ is a Lie superalgebra and the other is an associative superalgebra, we consider both of them as Lie superalgebras.

Remark 3.2.5. *(1) When $L_1 = L_2 = L$ and $\varphi = \mathrm{id}$, D is a superderivation of L.*

(2) A superderivation from a Lie superalgebra L to an associative superalgebra A will induce a same-parity superderivation from $U(L)$ to A.

(3) Let A be an associative superalgebra. Then a superderivation D of A as an associative superalgebra is also a superderivation of A as a Lie superalgebra.

(4) Let L_1 be generated by a subset S. Then a linear map $D : L_1 \to L_2$ satisfying (3.4) for all $u, v \in S$ can be extended uniquely, through Leibniz's rule, to a superderivation from L_1 to L_2, i.e., a superderivation is completely determined by its value on a generating subset.

3.2.1 Semi-infinite structure

Let $L = \bigoplus_{n\in\mathbb{Z}} L_n$ be a quasi-finite $\mathbb{Z}$-graded Lie algebra, with subalgebras $L_{\leq 0} = \bigoplus_{n\leq 0} L_n$ and $L_+ = \bigoplus_{n>0} L_n$. Let $\{e_i \mid i \leq 0\}$ and $\{e_i \mid i > 0\}$ be homogeneous bases of $L_{\leq 0}$ and L_+, respectively. Homogeneous means that each $e_i \in L_m$ for some $m \in \mathbb{Z}$. We also require that whenever $e_i \in L_m$, we have $e_{i+1} \in L_m$ or $e_{i+1} \in L_{m+1}$. Let $L^* = \bigoplus_{n\in\mathbb{Z}} L_n^*$ be the restricted dual of L with dual basis $\{e_i^* \mid i \in \mathbb{Z}\}$ such that $\langle e_i^*, e_j\rangle = \delta_{i,j}$, where $L_n^* := \mathrm{Hom}_{\mathbb{C}}(L_{-n}, \mathbb{C})$.

Definition 3.2.6. The space $\Lambda^{\infty/2+\bullet}L^*$ of *semi-infinite forms* on L is the vector space spanned by infinite wedge products of L^*, i.e.,

$$\omega = e^*_{i_1} \wedge e^*_{i_2} \wedge \cdots$$

for which there exists an integer $N(\omega)$ such that for all $k > N(\omega)$, we have $i_{k+1} = i_k - 1$.

Let $\iota(L)$ and $\varepsilon(L^*)$ be copies of L and L^*, with bases $\{\iota(e_i) \mid i \in \mathbb{Z}\}$ and $\{\varepsilon(e^*_i) \mid i \in \mathbb{Z}\}$, respectively. For $x \in L$ and $y^* \in L^*$, we denote by $\iota(x)$ and $\varepsilon(y^*)$ the corresponding elements in $\iota(L)$ and $\varepsilon(L^*)$, respectively. Define a Lie superalgebra

$$cl(L) := \iota(L) \oplus \varepsilon(L^*) \oplus \mathbb{C}K$$

with $\iota(L) \oplus \varepsilon(L^*)$ being odd (note that we assume that L is a Lie algebra, hence a purely even space), K being even, and with Lie superbracket: for $x, y \in L$ and $u^*, v^* \in L^*$,

$$[\iota(x), \iota(y)] = [\varepsilon(u^*), \varepsilon(v^*)] = 0, \quad [\iota(x), \varepsilon(u^*)] = \langle u^*, x\rangle K, \quad [K, cl(L)] = 0.$$

Note that $cl(L)$ inherits a natural $\mathbb{Z}$-grading from L with

$$cl(L)_n = \begin{cases} \iota(L_n) \oplus \epsilon(L^*_n) & \text{if } n \neq 0, \\ \iota(L_0) \oplus \epsilon(L^*_0) \oplus \mathbb{C}K & \text{if } n = 0. \end{cases}$$

By the definition of L^*, we have $\iota(e_i) \in cl(L)_n$ and $\varepsilon(e^*_i) \in cl(L)_{-n}$ when $e_i \in L_n$. The Lie superalgebra $cl(L)$ acts on $\Lambda^{\infty/2+\bullet}L^*$ in the following way, K acts as identity, and for $e_{i_0} \in L$,

$$\begin{aligned} \varepsilon(e^*_{i_0}) \cdot e^*_{i_1} \wedge e^*_{i_2} \wedge \cdots &= e^*_{i_0} \wedge e^*_{i_1} \wedge e^*_{i_2} \wedge \cdots, \\ \iota(e_{i_0}) \cdot e^*_{i_1} \wedge e^*_{i_2} \wedge \cdots &= \sum_{k \geq 1} (-1)^{k-1} \langle e^*_{i_k}, e_{i_0}\rangle e^*_{i_1} \wedge \cdots \wedge \hat{e}^*_{i_k} \wedge \cdots. \end{aligned}$$

The Clifford algebra $Cl(L \oplus L^*)$ is defined to be the quotient of $U(cl(L))$ by the ideal generated by $K - 1$, and it also has a well-defined action on $\Lambda^{\infty/2+\bullet}L^*$.

For a subspace V of L, we let $V^\perp = \{w^* \in L^* \mid \langle w^*, u\rangle = 0, \text{ for all } u \in V\}$. Then $L_+^\perp = \bigoplus_{n \geq 0} L^*_n$. Let $\omega_0 = e^*_0 \wedge e^*_{-1} \wedge e^*_{-2} \wedge \cdots$. Then

$$\iota(v) \cdot \omega_0 = \varepsilon(u^*) \cdot \omega_0 = 0, \text{ for } v \in L_+ \text{ and } u^* \in L_+^\perp. \tag{3.5}$$

The elements $\iota(v), \varepsilon(u^*)$ with $v \in L_+$ and $u^* \in L_+^\perp$ are called *annihilation operators*. Note that two annihilation operators always anticommute with each other. One can show that the space of semi-infinite forms $\Lambda^{\infty/2+\bullet}L^*$ on L is the irreducible Fock module of $Cl(L \oplus L^*)$ generated by the "vacuum" vector ω_0, with relations defined by (3.5). Every element of $\Lambda^{\infty/2+\bullet}L^*$ can be written as a linear combination of monomials of the form

$$\iota(e_{i_1}) \cdots \iota(e_{i_s}) \varepsilon(e^*_{j_1}) \cdots \varepsilon(e^*_{j_t}) \cdot \omega_0.$$

Remark 3.2.7. *Note that (3.5) implies that $cl(L)_n \cdot \omega_0 = 0$ for $n > 0$. In particular, $\Lambda^{\infty/2+\bullet}L^*$ is a smooth $cl(L)$-module on which K acts as identity, and the action can be extended to $U_1(cl(L))^{com} := U(cl(L))^{com}/(K-1)$.*

We want to define an L-action on $\Lambda^{\infty/2+\bullet}L^*$ through that of $cl(L)$. For the moment we just call it an action, but not necessarily a Lie algebra action. For $x \in L_n$ with $n \neq 0$, we denote by $\rho(x)$, the action of x on $\Lambda^{\infty/2+\bullet}L^*$ defined by

$$\rho(x) \cdot e_{i_1}^* \wedge e_{i_2}^* \wedge \cdots := \sum_{k \geq 1} e_{i_1}^* \wedge \cdots \wedge \mathrm{ad}^* x(e_{i_k}^*) \wedge \cdots , \tag{3.6}$$

where ad^* is the coadjoint action of L on L^*. The above sum is finite, thanks to the definition of semi-infinite forms and the fact that $x \in L_n$ for some $n \neq 0$. It is easy to verify the following relations (as operators on $\Lambda^{\infty/2+\bullet}L^*$): for all $y \in L, z^* \in L^*$,

$$[\rho(x), \iota(y)] = \iota(\mathrm{ad}\, x(y)), \qquad [\rho(x), \varepsilon(z^*)] = \varepsilon(\mathrm{ad}^* x(z^*)). \tag{3.7}$$

For $x \in L_0$, we cannot use (3.6) because it may involve an infinite sum. Let $\omega_0 = e_0^* \wedge e_{-1}^* \wedge e_{-2}^* \wedge \cdots$, and choose $\beta \in L_0^*$, considered as a function on L such that $\beta(L_n) = 0$ for all $n \neq 0$. Define $\rho(x) \cdot \omega_0 := \beta(x)\omega_0$, and extend it to an action on $\Lambda^{\infty/2+\bullet}L^*$ by requiring (3.7). This can be done because $\Lambda^{\infty/2+\bullet}L^*$ is irreducible and generated by ω_0 as a module of the Clifford algebra $Cl(L \oplus L^*)$.

To give an explicit expression of the action $\rho(x)$, we define the *normal ordering* of two elements of $\iota(L) \oplus \varepsilon(L^*)$ as follows,

$$:\iota(e_i)\iota(e_j) := \iota(e_i)\iota(e_j), \ :\varepsilon(e_i^*)\varepsilon(e_j^*) := \varepsilon(e_i^*)\varepsilon(e_j^*), \text{ for all } \ i, j \in \mathbb{Z},$$

$$-:\varepsilon(e_j^*)\iota(e_i) :=: \iota(e_i)\varepsilon(e_j^*) := \begin{cases} \iota(e_i)\varepsilon(e_j^*) & \text{if } i \neq j \text{ or } i = j \leq 0, \\ -\varepsilon(e_j^*)\iota(e_i) & \text{if } i = j > 0. \end{cases}$$

Remark 3.2.8. *The idea of normal ordering is to make sure that annihilation operators always appear on the right side of a product. Given a product of multiple operators, for example, $w = \iota(e_{i_1})\varepsilon(e_{j_1}) \cdots \iota(e_{i_s})$, the normal ordering $:w:$ means that we should move the annihilation operators to the right side and then add the sign of the permutation for doing so.*

Thanks to normal ordering, for all $x \in L$, the following elements are well-defined in $U_1(cl(L))^{com}$ and we have

$$\sum_{i \in \mathbb{Z}} :\varepsilon(\mathrm{ad}^* x(e_i^*))\iota(e_i) := \sum_{i \in \mathbb{Z}} :\iota(\mathrm{ad}\, x(e_i))\varepsilon(e_i^*): .$$

Let

$$\rho^\beta(x) := \sum_{i \in \mathbb{Z}} :\iota(\mathrm{ad}\, x(e_i))\varepsilon(e_i^*): + \beta(x). \tag{3.8}$$

Then $\rho^\beta(x)$ has a well-defined action on $\Lambda^{\infty/2+\bullet}L^*$ as it is a smooth $cl(L)$-module. Moreover, $\rho^\beta(x)$ satisfies (3.7), i.e., for $y \in L$ and $z^* \in L^*$, we have

$$[\rho^\beta(x), \iota(y)] = \iota(\mathrm{ad}\, x(y)), \qquad [\rho^\beta(x), \varepsilon(z^*)] = \varepsilon(\mathrm{ad}^* x(z^*)). \tag{3.9}$$

Lemma 3.2.9. *The operator $\rho^\beta(x)$ realizes the action of $\rho(x)$ on $\Lambda^{\infty/2+\bullet}L^*$.*

Proof. Since both $\rho^\beta(x)$ and $\rho(x)$ satisfy (3.7), and $\Lambda^{\infty/2+\bullet}L^*$ is generated by $\omega_0 = e_0^* \wedge e_{-1}^* \wedge e_{-2}^* \wedge \cdots$ as a $Cl(L \oplus L^*)$-module, we only need to show that their actions on ω_0 coincide. For simplicity, we assume that $x = e_{i_x}$. By definition

$$\rho(e_{i_x}) \cdot \omega_0 = \begin{cases} \beta(e_{i_x})\omega_0 & \text{if } e_{i_x} \in L_0, \\ \sum_{k \geq 0} e_0^* \wedge \cdots \wedge \mathrm{ad}^* e_{i_x}(e_{-k}^*) \wedge \cdots & \text{if } e_{i_x} \in L_n \text{ and } n \neq 0. \end{cases}$$

Now let us calculate the action of $\rho^\beta(e_{i_x})$ on ω_0. When $e_{i_x} \in L_0$, there is an annihilation operator in each summand $: \iota(\mathrm{ad}\, e_{i_x}(e_i))\varepsilon(e_i^*) :$ since $[L_0, L_n] \subseteq L_n$. Therefore, the sum $\sum_i : \iota(\mathrm{ad}\, e_{i_x}(e_i))\varepsilon(e_i^*) :$ acts as zero on ω_0 and $\rho^\beta(e_{i_x}) \cdot \omega_0 = \beta(e_{i_x})\omega_0$. When $e_{i_x} \in L_n$ for some $n \neq 0$, we have $\beta(e_{i_x}) = 0$. Moreover, $\varepsilon(\mathrm{ad}^* e_{i_x}(e_i^*))$ always anticommutes with $\iota(e_i)$ as $[L_n, L_m] \subseteq L_{m+n}$, so we can drop :: in $\rho^\beta(e_{i_x})$. Remember that $\iota(e_i) \cdot \omega_0 = 0$ for all $i > 0$, so

$$\begin{aligned} \rho^\beta(e_{i_x}) \cdot \omega_0 &= \sum_{i \leq 0} \varepsilon(\mathrm{ad}^* e_{i_x}(e_i^*)) \cdot (-1)^i e_0^* \wedge \cdots \wedge \hat{e}_i^* \wedge \cdots \\ &= \sum_{i \leq 0} e_0^* \wedge \cdots \wedge \mathrm{ad}^* e_{i_x}(e_i^*) \wedge \cdots . \end{aligned}$$

□

One can show that the centers of the Clifford algebra $Cl(L \oplus L^*)$ and its completion $U_1(cl(L))^{com}$ are both trivial, i.e., they only contain the constants.

For $x, y \in L$, define

$$\gamma^\beta(x, y) := [\rho^\beta(x), \rho^\beta(y)] - \rho^\beta([x, y]). \tag{3.10}$$

It is clear that $\Lambda^{\infty/2+\bullet}L^*$ admits an L-module structure under $\rho^\beta(x)$ if and only if $\gamma^\beta(x, y) = 0$ for all $x, y \in L$. One can show that $\gamma^\beta(x, y)$ is central hence a constant in $U_1(cl(L))^{com}$. Indeed, it is a 2-cocycle, i.e.,

$$\gamma^\beta(x, [y, z]) + \gamma^\beta(y, [z, x]) + \gamma^\beta(z, [y, x]) = 0 \text{ for all } \quad x, y, z \in L.$$

Moreover, one can show that $\gamma^\beta(L_m, L_n) = 0$ whenever $m + n \neq 0$ [Vor93].

Definition 3.2.10. We say that L admits a *semi-infinite structure* through ρ^β if $\gamma^\beta(\cdot, \cdot) \equiv 0$, i.e., if $\Lambda^{\infty/2+\bullet}L^*$ is an L-module under the action $\rho^\beta(x)$. We say that L admits a semi-infinite structure if L admits a semi-infinite structure through ρ^β for some $\beta \in L_0^*$.

Remark 3.2.11. *We can drop the restriction that $\beta \in L_0^*$ for a more general definition. In Chapter 4, when we realize affine W-algebras as semi-infinite cohomology, we are in the more general case. But for the existence of a semi-infinite structure, the part which belongs to L_0^* is essential. For example, let $\beta = \sum_i \beta_i \in L^*$ with $\beta_i \in L_i^*$. Then ρ^β gives L a semi-infinite structure if and only if ρ^{β_0} does and $\partial\beta_i = 0$ for all $i \neq 0$. Here $\partial\beta_i(x, y) := \beta_i([x, y])$ for $x, y \in L$.*

Example 3.2.12. If L is abelian, it always admits a semi-infinite structure. When $H^2(L, \mathbb{C}) = 0$, every 2-cocycle is a coboundary. If $\gamma^\beta(\cdot, \cdot) \neq 0$, we can choose some $\beta' \in L^*$ (by [Vor93], we can choose $\beta' \in L_0^*$), such that $\partial\beta' = \gamma^\beta(\cdot, \cdot)$, then $\rho^{\beta-\beta'}$ gives a semi-infinite structure for L. For example, affine Kac-Moody algebras and the Virasoro algebra admit semi-infinite structures.

Let $\mathfrak{a}$ be a finite-dimensional Lie algebra. Recall that the affinization of $\mathfrak{a}$ is the tensor product $\hat{\mathfrak{a}} := \mathfrak{a} \otimes \mathbb{C}[t, t^{-1}]$ with Lie bracket: $[a \otimes t^n, b \otimes t^m] = [a, b] \otimes t^{m+n}$ for all $a, b \in \mathfrak{a}$ and $m, n \in \mathbb{Z}$, where $\mathbb{C}[t, t^{-1}]$ is the ring of Laurent polynomials. It has a natural $\mathbb{Z}$-grading with $\hat{\mathfrak{a}}_n := \mathfrak{a} \otimes t^n$.

Proposition 3.2.13. *Let $\mathfrak{n}$ be a finite-dimensional nilpotent Lie algebra. Then $\hat{\mathfrak{n}}$ admits a semi-infinite structure.*

Proof. Let $\dim \mathfrak{n} = d$ and $\mathcal{B} := \{e_i\}_{1 \leq i \leq d}$ be a basis of $\mathfrak{n}$, with structure constants $\{c_{i,j}^k\}$ such that $[e_i, e_j] = \sum_{k=1}^d c_{i,j}^k e_k$. Since $\mathfrak{n}$ is nilpotent, by Engel's theorem, we can choose the basis $\mathcal{B}$, such that $c_{i,j}^k = 0$ for $k \geq j$. In the language of matrices, $\operatorname{ad} e_i \in \mathfrak{gl}(\mathfrak{n})$ with respect to $\mathcal{B}$ are strictly upper triangular matrices for all i. In particular, we have $c_{i,j}^j = 0$. We fix such a basis $\mathcal{B}$, and let $\{e_i^*\}_{1 \leq i \leq d}$ be the dual basis of $\mathfrak{n}^*$. Identify the restricted dual $\hat{\mathfrak{n}}^*$ of $\hat{\mathfrak{n}}$ with $\mathfrak{n}^* \otimes \mathbb{C}[t, t^{-1}]$ through the pairing $\langle e_j^* \otimes t^m, e_i \otimes t^n \rangle = \delta_{n,-m}\delta_{i,j}$. For convenience, we denote by $e_{i,n} := e_i \otimes t^n$ and $e_{i,n}^* := e_i^* \otimes t^{-n}$. Then $\{e_{i,n}\}$ and $\{e_{i,n}^*\}$ form dual bases of $\hat{\mathfrak{n}}$ and $\hat{\mathfrak{n}}^*$, respectively. The adjoint action gives

$$\operatorname{ad} e_{i,n}(e_{j,m}) = [e_{i,n}, e_{j,m}] = [e_i, e_j] \otimes t^{m+n} = \sum_{k=1}^d c_{i,j}^k e_{k,m+n}.$$

For the coadjoint action, we have $\operatorname{ad}^* e_{i,n}(e_{j,m}^*) = \sum_{k=1}^d c_{k,i}^j e_{k,m-n}^*$.

Let

$$\rho^0(x) = \sum_{\substack{i=1,\cdots,d,\\ n \in \mathbb{Z}}} :\iota(\operatorname{ad} x(e_{i,n}))\varepsilon(e_{i,n}^*): .$$

We show that $\gamma^0(x, y) := [\rho^0(x), \rho^0(y)] - \rho^0([x, y]) = 0$ for all $x, y \in \hat{\mathfrak{n}}$, i.e., $\hat{\mathfrak{n}}$ admits a semi-infinite structure through ρ^0.

For simplicity, assume that $x = e_{i_x,n_x}$ and $y = e_{i_y,n_y}$. Since $\iota(\operatorname{ad} x(e_{i,n}))$ anticommutes with $\varepsilon(e_{i,n}^*)$

by the choice of basis of $\mathfrak{n}$, we can drop the normal ordering :: in $\rho^0(x)$, so we have

$$\begin{aligned}
&[\rho^0(e_{i_x,n_x}), \rho^0(e_{i_y,n_y})] \\
&\quad = \sum_{\substack{i,j=1,\cdots,d,\\ m,n\in\mathbb{Z}}} \left[\iota(\mathrm{ad}\, e_{i_x,n_x}(e_{i,n}))\varepsilon(e^*_{i,n}),\, \varepsilon(\mathrm{ad}^* e_{i_y,n_y}(e^*_{j,m}))\iota(e_{j,m})\right] \\
&\quad = A + B,
\end{aligned}$$

where

$$\begin{aligned}
A &= \sum_{\substack{i,j=1,\cdots,d,\\ m,n\in\mathbb{Z}}} \iota(\mathrm{ad}\, e_{i_x,n_x}(e_{i,n})) \left[\varepsilon(e^*_{i,n}), \varepsilon(\mathrm{ad}^* e_{i_y,n_y}(e^*_{j,m}))\iota(e_{j,m})\right], \\
B &= \sum_{\substack{i,j=1,\cdots,d,\\ m,n\in\mathbb{Z}}} \left[\iota(\mathrm{ad}\, e_{i_x,n_x}(e_{i,n})), \varepsilon(\mathrm{ad}^* e_{i_y,n_y}(e^*_{j,m}))\iota(e_{j,m})\right] \varepsilon(e^*_{i,n}).
\end{aligned}$$

Note that

$$\begin{aligned}
A &= -\sum_{\substack{i=1,\cdots,d,\\ n\in\mathbb{Z}}} \iota(\mathrm{ad}\, e_{i_x,n_x}(e_{i,n}))\varepsilon(\mathrm{ad}^* e_{i_y,n_y}(e^*_{i,n})) \\
&= -\sum_{\substack{i,j,k=1,\cdots,d,\\ n\in\mathbb{Z}}} c^j_{i_x,i} c^i_{k,i_y} \iota(e_{j,n+n_x})\varepsilon(e^*_{k,n-n_y}),
\end{aligned}$$

and

$$\begin{aligned}
B &= \sum_{\substack{i,j=1,\cdots,d,\\ m,n\in\mathbb{Z}}} \langle \mathrm{ad}^* e_{i_y,n_y}(e^*_{j,m}), \mathrm{ad}\, e_{i_x,n_x}(e_{i,n})\rangle \iota(e_{j,m})\varepsilon(e^*_{i,n}) \\
&= \sum_{\substack{i,j,k=1,\cdots,d,\\ n\in\mathbb{Z}}} c^j_{k,i_y} c^k_{i_x,i} \iota(e_{j,n+n_x+n_y})\varepsilon(e^*_{i,n}) \\
&= \sum_{\substack{i,j,k=1,\cdots,d,\\ m\in\mathbb{Z}}} c^j_{k,i_y} c^k_{i_x,i} \iota(e_{j,m+n_x})\varepsilon(e^*_{i,m-n_y}).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\rho^0([e_{i_x,n_x}, e_{i_y,n_y}]) &= \sum_{\substack{i=1,\cdots,d,\\ n\in\mathbb{Z}}} \iota(\mathrm{ad}\, [e_{i_x,n_x}, e_{i_y,n_y}](e_{i,n}))\varepsilon(e^*_{i,n}) \\
&= \sum_{\substack{i,j,k=1,\cdots,d,\\ n\in\mathbb{Z}}} c^j_{i_x,i_y} c^k_{j,i} \iota(e_{k,n+n_x+n_y})\varepsilon(e^*_{i,n}) \\
&= \sum_{\substack{i,j,k=1,\cdots,d,\\ m\in\mathbb{Z}}} c^j_{i_x,i_y} c^k_{j,i} \iota(e_{k,m+n_x})\varepsilon(e^*_{i,m-n_y}).
\end{aligned}$$

Now $[\rho^0(x), \rho^0(y)] - \rho^0([x,y]) = 0$ comes from the Jacobi identity of the structure constants,

$$-\sum_i c^j_{i_x,i} c^i_{k,i_y} + \sum_i c^j_{i,i_y} c^i_{i_x,k} = \sum_i c^i_{i_x,i_y} c^j_{i,k}.$$

□

3.2.2 Semi-infinite cohomology

In this subsection, we assume that L is a quasi-finite $\mathbb{Z}$-graded Lie algebra admitting a semi-infinite structure through ρ^β defined by (3.8), i.e., $\gamma^\beta(\cdot,\cdot)\equiv 0$ and the map $\rho^\beta : L \to U_1(cl(L))^{com}$ defined by $x \mapsto \rho^\beta(x)$ is a Lie algebra homomorphism, which gives $\Lambda^{\infty/2+\bullet}L^*$ an L-module structure.

Let $\theta^\beta : L \to U(L)\otimes U_1(cl(L))^{com}$ be the map defined by

$$\theta^\beta(x) := x + \rho^\beta(x). \tag{3.11}$$

Remark 3.2.14. *Note that we omitted the tensor product $\otimes$ in (3.11), so $\theta^\beta(x) = x\otimes 1 + 1\otimes\rho^\beta(x)$. We will use the same notation in the sequel.*

The map θ^β is obviously a Lie algebra homomorphism. Let M be a smooth L-module. Then the tensor product $M\otimes\Lambda^{\infty/2+\bullet}L^*$ is naturally a $U(L)\otimes U_1(cl(L))^{com}$-module hence a smooth L-module under the action $\theta^\beta(x)$. Since x commutes with $\iota(L)$ and $\varepsilon(L^*)$, we have: for all $y\in L, z^*\in L^*$,

$$[\theta^\beta(x),\iota(y)] = \iota([x,y]), \qquad [\theta^\beta(x),\varepsilon(z^*)] = \varepsilon(\mathrm{ad}^* x(z^*)).$$

Let

$$\begin{aligned} d^\beta &= \sum_{i\in\mathbb{Z}} e_i\varepsilon(e_i^*) - \sum_{i<j} :\iota([e_i,e_j])\varepsilon(e_i^*)\varepsilon(e_j^*): + \varepsilon(\beta) \\ &= \sum_{i\in\mathbb{Z}} e_i\varepsilon(e_i^*) - \frac{1}{2}\sum_{i,j\in\mathbb{Z}} :\iota([e_i,e_j])\varepsilon(e_i^*)\varepsilon(e_j^*): + \varepsilon(\beta). \end{aligned} \tag{3.12}$$

Then $d^\beta \in U(L)^{com}\otimes U_1(cl(L))^{com}$ has a well-defined action on $M\otimes\Lambda^{\infty/2+\bullet}L^*$.

Lemma 3.2.15. *We have $[d^\beta,\iota(x)] = \theta^\beta(x)$ for all $x\in L$.*

Proof. For simplicity, we assume that $x = e_k$ for some $k\in\mathbb{Z}$. Then

$$\left[\sum_{i\in\mathbb{Z}} e_i\varepsilon(e_i^*) + \varepsilon(\beta), \iota(e_k)\right] = e_k + \beta(e_k),$$

and

$$\begin{aligned} &\left[-\sum_{i<j} :\iota([e_i,e_j])\varepsilon(e_i^*)\varepsilon(e_j^*):, \iota(e_k)\right] \\ &\quad = -\sum_{i<k} :\iota([e_i,e_k])\varepsilon(e_i^*): + \sum_{k<j} :\iota([e_k,e_j])\varepsilon(e_j^*): \\ &\quad = \sum_{i\in\mathbb{Z}} :\iota(\mathrm{ad}\, e_k(e_i))\varepsilon(e_i^*):. \end{aligned}$$

Therefore, we have $[d^\beta,\iota(e_k)] = \theta^\beta(e_k)$. ∎

We define a charge grading on $cl(L)$ by setting

$$-\mathrm{cdeg}\,\iota(x) = \mathrm{cdeg}\,\varepsilon(y^*) = 1 \quad \text{for } x \in L, y^* \in L^*, \quad \text{and} \quad \mathrm{cdeg}\, K = 0. \tag{3.13}$$

When we refer to the charge gradation, we will add the superscript $^{⁎}$. We have

$$cl(L) = cl(L)^{⁎}_{-1} \oplus cl(L)^{⁎}_{0} \oplus cl(L)^{⁎}_{1}$$

with $cl(L)^{⁎}_1 = \varepsilon(L^*), cl(L)^{⁎}_0 = \mathbb{C}K$ and $cl(L)^{⁎}_{-1} = \iota(L)$. This induces a charge gradation on $U(cl(L))$ and also on the Clifford algebra $Cl(L \oplus L^*)$. As a simple module of $Cl(L \oplus L^*)$, the space of semi-infinite forms $\Lambda^{\infty/2+\bullet}L^*$ inherits a charge gradation if we set $\mathrm{cdeg}\,\omega_0 = 0$, with

$$\Lambda^{\infty/2+n}L^* := (\Lambda^{\infty/2+\bullet}L^*)^{⁎}_n = \mathrm{span}_{\mathbb{C}}\{\iota(e_{i_1})\cdots\iota(e_{i_s})\varepsilon(e^*_{j_1})\cdots\varepsilon(e^*_{j_t})\cdot\omega_0 \mid t-s=n\}.$$

With respect to the charge gradation, the operator $\rho^\beta(x)$ is of degree zero for all $x \in L$, so each component $\Lambda^{\infty/2+n}L^*$ is an L-submodule. If we define the charge degree of M to be zero, then d^β is a charge degree 1 operator on $M \otimes \Lambda^{\infty/2+\bullet}L^*$.

Proposition 3.2.16 ([Vor93], Proposition 2.6)**.** *The operator d^β does not depend on the choice of basis of L, and $(d^\beta)^2 = 0$.*

Definition 3.2.17. The complex $(M \otimes \Lambda^{\infty/2+\bullet}L^*, d^\beta)$ is called the *Feigin standard complex* and its cohomology $H^{\infty/2+\bullet}(L, \beta, M)$ the *semi-infinite cohomology* of L with coefficients in M. When $\beta = 0$, we write just as $H^{\infty/2+\bullet}(L, M)$.

Remark 3.2.18. *There is an interesting characterization of the differential d^β in [Akm93] and in [Ara17] for affine W-algebras in the principal nilpotent cases, which can be realized as a semi-infinite cohomology. To contrast with our adjusted version in the next section, we will also call the cohomology in Definition 3.2.17 ordinary semi-infinite cohomology.*

We write β in the cohomology because it plays some role. Indeed, if $\rho^{\beta'}$ gives another semi-infinite structure, one can show that $(\beta - \beta')([L, L]) = 0$, so $\beta - \beta'$ defines a 1-dimensional module $\mathbb{C}_{\beta-\beta'}$ of L, on which $x \in L$ acts as $(\beta - \beta')(x)$.

Proposition 3.2.19 ([Vor93], Proposition 2.7)**.** *If both ρ^β and $\rho^{\beta'}$ give semi-infinite structures on L, then*

$$H^{\infty/2+\bullet}(L, \beta, M) \cong H^{\infty/2+\bullet}(L, \beta', M \otimes \mathbb{C}_{\beta-\beta'}).$$

3.3 An adjustment when the 2-cocycle $\gamma^\beta(\cdot,\cdot)$ is not identically zero

Recall the notation in the previous section. We assume that $\gamma^\beta(\cdot,\cdot)$ is not identically zero in this section, i.e., ρ^β does not give a semi-infinite structure on L.

3.3.1 What is the problem

Let d^β be the operator defined by (3.12) and let us consider the value $[[(d^\beta)^2, \iota(x)], \iota(y)]$ for $x, y \in L$. Since d^β is odd, we have $(d^\beta)^2 = \dfrac{1}{2}[d^\beta, d^\beta]$, hence $[(d^\beta)^2, \iota(x)] = [d^\beta, [d^\beta, \iota(x)]]$. By Lemma 3.2.15, we have $[d^\beta, \iota(x)] = \theta^\beta(x)$ (though we assume that $\gamma^\beta(\cdot,\cdot) \equiv 0$ in that section, the calculations in Lemma 3.2.15 still hold), so

$$\begin{aligned}
[[(d^\beta)^2, \iota(x)], \iota(y)] &= [[d^\beta, \theta^\beta(x)], \iota(y)] \\
&= [d^\beta, [\theta^\beta(x), \iota(y)]] + [[d^\beta, \iota(y)], \theta^\beta(x)] \\
&= [d^\beta, \iota([x,y])] + [\theta^\beta(y), \theta^\beta(x)] \\
&= \theta^\beta([x,y]) - [\theta^\beta(x), \theta^\beta(y)] \\
&= -\gamma^\beta(x,y). \qquad (3.14)
\end{aligned}$$

In particular, the operator d^β is not of square zero if $\gamma^\beta(\cdot,\cdot)$ is not identically zero.

Let $\ker \gamma^\beta := \{x \in L \mid \gamma(x, L) \equiv 0\}$ be the radical of the 2-cocycle $\gamma^\beta(\cdot,\cdot)$. Then $\ker \gamma^\beta$ is obviously a graded subalgebra of L. Let us choose a graded complement of $\ker \gamma^\beta$ in L, which we denote by F_β. Then $L = \ker \gamma^\beta \oplus F_\beta$, and $\gamma^\beta(\cdot,\cdot)$ is non-degenerate on F_β. Let $\epsilon(F_\beta)$ be a copy of F_β. For $x \in L$, we use $\epsilon(x)$ to denote its projection in F_β but considered as an element of $\epsilon(F_\beta)$. Then $\epsilon(\ker \gamma^\beta) = 0$.

Consider the Lie superalgebra

$$c(L) := \iota(L) \oplus \varepsilon(L^*) \oplus \mathbb{C}K \oplus \epsilon(F_\beta),$$

which contains $cl(L)$ as a subalgebra. By definition, the subspace $\epsilon(F_\beta)$ is even, commutes with $cl(L)$, and has bracket: $[\epsilon(x), \epsilon(y)] = -\gamma^\beta(x,y)K$ for $x, y \in F_\beta$. Since F_β is a graded subspace of L, the subalgebra $\epsilon(F_\beta) \oplus \mathbb{C}K$ is $\mathbb{Z}$-graded with

$$(\epsilon(F_\beta) \oplus \mathbb{C}K)_n = \begin{cases} \epsilon((F_\beta)_n) & \text{if } n \neq 0, \\ \epsilon((F_\beta)_0) \oplus \mathbb{C}K & \text{if } n = 0. \end{cases}$$

The subspace $\epsilon(F_\beta)_+ := \left(\bigoplus_{n>0} \epsilon(F_\beta)_n\right) \oplus \mathbb{C}K$ is an abelian subalgebra, thanks to the property that $\gamma^\beta(L_m, L_n) \equiv 0$ if $m + n \neq 0$. Let $\mathbb{C}$ be the 1-dimensional module of this abelian subalgebra on which $\bigoplus_{n>0} \epsilon(F_\beta)_n$ acts as zero and K acts as the identity. We call the induced module

$$\mathfrak{F}_\beta = \mathrm{Ind}_{\epsilon(F_\beta)_+}^{\epsilon(F_\beta)\oplus\mathbb{C}K} \mathbb{C} \qquad (3.15)$$

the *Fock representation* of $\epsilon(F_\beta) \oplus \mathbb{C}K$, which is obviously smooth. Remember that $\Lambda^{\infty/2+\bullet}L^*$ is a smooth $cl(L)$-module on which K also acts as identity, so $\Lambda^{\infty/2+\bullet}L^* \otimes \mathfrak{F}_\beta$ is a smooth $c(L)$-module.

Let $U_1(c(L))^{com} := U(c(L))^{com}/(K-1)$, and define a map $\bar{\rho}^\beta : L \to U_1(c(L))^{com}$ by

$$\bar{\rho}^\beta(x) := \rho^\beta(x) + \epsilon(x).$$

Then $\bar{\rho}^\beta(x)$ has a well-defined action on $\Lambda^{\infty/2+\bullet}L^* \otimes \mathfrak{F}_\beta$, and for $x, y \in L, z^* \in L^*$, we have

$$[\bar{\rho}^\beta(x), \iota(y)] = \iota([x,y]), \quad [\bar{\rho}^\beta(x), \varepsilon(z^*)] = \varepsilon(\mathrm{ad}^* x(z^*)), \quad [\bar{\rho}^\beta(x), \epsilon(y)] = -\gamma^\beta(x,y). \tag{3.16}$$

Let $s(L) = L \oplus c(L)$ be the direct sum of L and $c(L)$. Then $s(L)$ inherits a natural $\mathbb{Z}$-grading from L and $c(L)$. Let

$$U_1(s(L))^{com} := U(s(L))^{com}/(K-1) \cong U(L)^{com} \otimes U_1(c(L))^{com},$$

and

$$\bar{\theta}^\beta(x) = x + \bar{\rho}^\beta(x) \in U_1(s(L))^{com}. \tag{3.17}$$

Let M be a smooth L-module. Then $\bar{\theta}^\beta(x)$ has a well-defined action on $M \otimes \Lambda^{\infty/2+\bullet}L^* \otimes \mathfrak{F}_\beta$. We have $[\bar{\theta}^\beta(x), y] = [x,y]$ for all $x, y \in L$, moreover,

$$[\bar{\theta}^\beta(x), \iota(y)] = \iota([x,y]), \quad [\bar{\theta}^\beta(x), \varepsilon(z^*)] = \varepsilon(\mathrm{ad}^* x(z^*)), \quad [\bar{\theta}^\beta(x), \epsilon(y)] = -\gamma^\beta(x,y). \tag{3.18}$$

Lemma 3.3.1. *The map $\bar{\rho}^\beta : L \longrightarrow U_1(c(L))^{com}$ is a Lie algebra homomorphism if $[L,L] \subseteq \ker\,\gamma^\beta$.*

Proof. We need to prove $\bar{\rho}^\beta([x,y]) = [\bar{\rho}^\beta(x), \bar{\rho}^\beta(y)]$ for all $x, y \in L$. But we have

$$\begin{aligned}
[\bar{\rho}^\beta(x), \bar{\rho}^\beta(y)] &= [\rho^\beta(x) + \epsilon(x), \rho^\beta(y) + \epsilon(y)] \\
&= [\rho^\beta(x), \rho^\beta(y)] + [\epsilon(x), \epsilon(y)] \\
&= \rho^\beta([x,y]) + \gamma^\beta(x,y) - \gamma^\beta(x,y) \\
&= \rho^\beta([x,y])
\end{aligned}$$

and $\bar{\rho}^\beta([x,y]) = \rho^\beta([x,y])$ if $\epsilon([x,y]) \equiv 0$, i.e., if $[L,L] \subseteq \ker\,\gamma^\beta$. □

Remark 3.3.2. *Lemma 3.3.1 tells us that even though $\Lambda^{\infty/2+\bullet}L^*$ is not an L-module under the action $\rho^\beta(x)$, the tensor product $\Lambda^{\infty/2+\bullet}L^* \otimes \mathfrak{F}_\beta$ is under $\bar{\rho}^\beta(x)$.*

Assumption: From now on, we assume that $[L,L] \subseteq \ker\,\gamma^\beta$ is satisfied.

3.3.2 Construction and characterization of a square zero differential

We extend the charge gradation (see (3.13)) on $cl(L)$ to $c(L)$ by setting $\mathrm{cdeg}\,\epsilon(F_\beta) = 0$, and then to $s(L)$ by setting $\mathrm{cdeg}\,L = 0$. As usual, we denote the charge gradation by adding a superscript *. These charge gradations induce another $\mathbb{Z}$-grading on their universal enveloping algebras, which are different from those induced from the quasi-finite $\mathbb{Z}$-grading. At the module level, if we set $\mathrm{cdeg}\,M = \mathrm{cdeg}\,\mathfrak{F}_\beta = 0$ for a smooth L-module M, and the charge gradation on $\Lambda^{\infty/2+\bullet}L^*$ as before, then $\Lambda^{\infty/2+\bullet}L^* \otimes \mathfrak{F}_\beta$ is a $\mathbb{Z}$-graded $c(L)$-module and $M \otimes \Lambda^{\infty/2+\bullet}L^* \otimes \mathfrak{F}_\beta$ a $\mathbb{Z}$-graded $s(L)$-module under the charge gradations.

Let $i_c : c(L) \hookrightarrow U_1(c(L))^{com}$ and $i_s : s(L) \hookrightarrow U_1(s(L))^{com}$ be the canonical inclusions.

Definition 3.3.3. A superderivation D with respect to i_c or i_s, is said to be of *charge degree* N if $D(c(L)^{\circledast}_n) \subseteq U_1(c(L))^{com,\circledast}_{n+N}$ or $D(s(L)^{\circledast}_n) \subseteq U_1(s(L))^{com,\circledast}_{n+N}$, respectively. A superderivation D of $c(L)$ or of $s(L)$ is said to be of *charge degree* N if $D(c(L)^{\circledast}_n) \subseteq c(L)^{\circledast}_{n+N}$ or $D(s(L)^{\circledast}_n) \subseteq s(L)^{\circledast}_{n+N}$, respectively.

Define an action of L on $c(L)$ as follows. For $x, y \in L, z \in L^*$,

$$x \cdot \iota(y) = \iota([x, y]), \quad x \cdot \varepsilon(z^*) = \varepsilon(\mathrm{ad}^* x(z^*)), \quad x \cdot \epsilon(y) = -\gamma^\beta(x, y)K, \quad x \cdot K = 0.$$

We extend this action to $s(L)$ by letting L act on itself by the adjoint action.

Lemma 3.3.4. *The actions of $x \in L$ on $c(L)$ and $s(L)$ are even derivations of charge degree zero.*

Proof. This can be verified by direct calculations, as we know explicitly both the Lie brackets of $c(L), s(L)$ and the actions of L on them. They are obviously of charge degree zero. □

Remark 3.3.5. *The actions of x on $c(L)$ and $s(L)$ induce even derivations of charge degree zero on $U_1(c(L))^{com}$ and $U_1(s(L))^{com}$, respectively. The inner derivations $[\bar{\rho}^\beta(x), \cdot]$ and $[\bar{\theta}^\beta(x), \cdot]$ realize the actions of x on $U_1(c(L))^{com}$ and $U_1(s(L))^{com}$, respectively, by (3.16) and (3.18).*

Lemma 3.3.6. *Let $u \in U_1(s(L))^{com}$ be a charge degree ≥ 1 element. Then $[u, \iota(x)] = 0$ for all $x \in L$ only if $u = 0$.*

Proof. As $\mathrm{cdeg}\, u \geq 1$, if u is not zero, we can write

$$u = w\varepsilon(e_k^*) + v \qquad \text{or} \qquad u = \varepsilon(e_k^*)w + v$$

for some $k \in \mathbb{Z}$ with $w, v \in U_1(s(L))^{com}$ and $w \neq 0$, such that $\varepsilon(e_k^*)$ does not appear in w or v, i.e.,

$$[w, \iota(e_k)] = [v, \iota(e_k)] = 0.$$

Then $[u, \iota(e_k)] = w \neq 0$ gives a contradiction. □

Lemma 3.3.7. *Let D be a superderivation of charge degree ≥ 1 with respect to $i_s : s(L) \hookrightarrow U_1(s(L))^{com}$, and suppose that D kills K. Then D is determined by its value on $\iota(L)$.*

Proof. Since $s(L)$ is generated by $L \oplus \iota(L) \oplus \varepsilon(L^*) \oplus \epsilon(F_\beta)$, we just need to show that the value of D on $L \oplus \varepsilon(L^*) \oplus \epsilon(F_\beta)$ is determined by its value on $\iota(L)$. Let D' be another superderivation, such that D' kills K and coincide with D on $\iota(L)$. We show that $D = D'$. Since $D - D'$ is also a superderivation, we have

$$(D - D')[u, v] = [(D - D')u, v] + (-1)^{i \cdot p(u)}[u, (D - D')v] \tag{3.19}$$

for all $u, v \in s(L)$, where i is the parity of D and D'.

Note that $[s(L), \iota(L)] \subseteq \mathbb{C}K$ and $(D - D')K = (D - D')\iota(L) = 0$. Let $u \in s(L), v = \iota(x) \in \iota(L)$ in (3.19). Then we have

$$[(D - D')u, \iota(x)] = 0. \tag{3.20}$$

If $u \in \iota(L)$, then $(D-D')u = 0$. If $u \in L \oplus \epsilon(F_\beta) \oplus \varepsilon(L^*)$, then note that $\mathrm{cdeg}\,(D-D')u \geq 1$ when $u \in L \oplus \epsilon(F_\beta)$, and $\mathrm{cdeg}\,(D - D')u \geq 2$ when $u \in \varepsilon(L^*)$. Since (3.20) holds for all $\iota(x) \in \iota(L)$, Lemma 3.3.6 ensures that $(D - D')u = 0$, i.e., $D = D'$ on $s(L)$. □

Remark 3.3.8. *An equivalent statement of Lemma 3.3.7 is, given a charge degree ≥ 1 superderivation with respect to the inclusion $i_{\iota(L)} : \iota(L) \to U_1(s(L))^{com}$, we can extend it to be a superderivation of the same charge degree with respect to the inclusion $i_s : s(L) \to U_1(s(L))^{com}$ in a unique way.*

Recall that $\bar{\theta}^\beta(x)$ defined by (3.17) is even and satisfies (3.18), in particular, we have

$$[\bar{\theta}^\beta(x), \iota(y)] - [\iota(x), \bar{\theta}^\beta(y)] = \iota([x, y]) + \iota([y, x]) = 0.$$

As $\iota(L)$ is an abelian subalgebra of $s(L)$, the map $D : \iota(L) \to U_1(s(L))^{com}$ sending $\iota(x)$ to $\bar{\theta}^\beta(x)$ is an odd superderivation of charge degree 1 with respect to $i_{\iota(L)}$, so it can be extended to be a superderivation with respect to i_s in a unique way.

Let

$$\bar{d}^\beta = d^\beta + \sum_{i \in \mathbb{Z}} \varepsilon(e_i^*)\epsilon(e_i). \tag{3.21}$$

Theorem 3.3.9. *The element $\bar{d}^\beta$ defined by (3.21) is the unique element in $U_1(s(L))^{com}$ of charge degree 1, such that $[\bar{d}^\beta, \iota(x)] = \bar{\theta}^\beta(x)$ for all $x \in L$, and we have $(\bar{d}^\beta)^2 = 0$.*

Proof. By Lemma 3.2.15, we already have $[d^\beta, \iota(x)] = \theta^\beta(x)$, so we only need to show that

$$\sum_{i \in \mathbb{Z}} [\varepsilon(e_i^*)\epsilon(e_i), \iota(x)] = \epsilon(x).$$

This is obvious for $x = e_k$ hence true for all $x \in L$. The uniqueness is by Lemma 3.3.6.

The operators $[(\bar{d}^\beta)^2, \cdot]$ and $[[(\bar{d}^\beta)^2, \iota(x)], \cdot]$ are derivations of charge degree 2 and 1, respectively, if they are non-zero. By Lemma 3.3.7, they are completely determined by their value on $\iota(L)$. Recall the calculations in (3.14). Since $[L, L] \subseteq \ker\, \gamma^\beta$ and $[\bar{d}^\beta, \iota(x)] = \bar{\theta}^\beta(x)$, we have

$$\begin{aligned}
[[(\bar{d}^\beta)^2, \iota(x)], \iota(y)] &= \bar{\theta}^\beta([x, y]) - [\bar{\theta}^\beta(x), \bar{\theta}^\beta(y)] \\
&= \rho^\beta([x, y]) + [x, y] - [\rho^\beta(x) + x + \epsilon(x), \rho^\beta(y) + y + \epsilon(y)] \\
&= \rho^\beta([x, y]) - [\rho^\beta(x), \rho^\beta(y)] + \gamma^\beta(x, y) \\
&= 0,
\end{aligned}$$

for $x, y \in L$. Lemma 3.3.6 then implies that $[(\bar{d}^\beta)^2, \iota(x)] = 0$ for all $x \in L$ hence $(\bar{d}^\beta)^2 = 0$. □

Definition 3.3.10. We call the complex $(M \otimes \Lambda^{\infty/2+\bullet} L^* \otimes \mathfrak{F}_\beta, \bar{d}^\beta)$ the *adjusted Feigin complex* with respect to β, and its cohomology $H_a^{\infty/2+\bullet}(L, \beta, M)$ the *adjusted semi-infinite cohomology* of L with coefficients in M, with respect to β.

Remark 3.3.11. *Note that we used a subscript "a" in the adjusted semi-infinite cohomology.*

3.3.3 Comparison with ordinary semi-infinite cohomology

Our adjustment sometimes gives nothing new but ordinary semi-infinite cohomology with coefficients in another module. Assume that $\rho^\beta(x)$ gives a semi-infinite structure on L, and $\beta' \in \bigoplus_{n\geq 0} L_n^*$ is a 1-cochain[1] such that $\partial\beta' \neq 0$ but $\partial\beta'([L,L], L) = 0$, where $\partial\beta'(x,y) = \beta'([x,y])$. Then $\gamma^{\beta+\beta'} = -\partial\beta' \neq 0$ and $[L,L] \subseteq \ker \gamma^{\beta+\beta'}$. We can therefore talk about adjusted semi-infinite cohomology of L with coefficients in a smooth module M with respect to $\beta + \beta'$, which is the cohomology of the complex $(M \otimes \Lambda^{\infty/2+\bullet} \otimes \mathfrak{F}_{\beta+\beta'}, \bar{d}^{\beta+\beta'})$.

Recall that

$$\begin{aligned}\bar{d}^{\beta+\beta'} &= \sum_{i\in\mathbb{Z}} e_i \varepsilon(e_i^*) - \frac{1}{2}\sum_{i,j\in\mathbb{Z}} :\iota([e_i,e_j])\varepsilon(e_i^*)\varepsilon(e_j^*): + \varepsilon(\beta+\beta') + \sum_{i\in\mathbb{Z}} \varepsilon(e_i^*)\epsilon(e_i)\\ &= \sum_{i\in\mathbb{Z}} \varepsilon(e_i^*)(e_i + \beta'(e_i) + \epsilon(e_i)) - \frac{1}{2}\sum_{i,j\in\mathbb{Z}} :\iota([e_i,e_j])\varepsilon(e_i^*)\varepsilon(e_j^*): + \varepsilon(\beta),\end{aligned}$$

and

$$[\bar{d}^{\beta+\beta'}, \iota(x)] = x + \beta'(x) + \epsilon(x) + \rho^\beta(x).$$

On the other hand, since $[\epsilon(x), \epsilon(y)] = -\gamma^{\beta+\beta'}(x,y) = \beta'([x,y])$ and $\epsilon([x,y]) \equiv 0$, we have

$$[x + \beta'(x) + \epsilon(x), y + \beta'(y) + \epsilon(y)] = [x,y] + \beta'([x,y]),$$

that is, $M \otimes \mathfrak{F}_{\beta+\beta'}$ is an L-module under the action $x + \beta'(x) + \epsilon(x)$, and it is smooth. Therefore, we have the following theorem.

Theorem 3.3.12. *Let β, β' be as above. Then*

$$H_a^{\infty/2+\bullet}(L, \beta+\beta', M) \cong H^{\infty/2+\bullet}(L, \beta, M \otimes \mathfrak{F}_{\beta+\beta'}).$$

[1] We require that $\beta' \in \bigoplus_{n\geq 0} L_n^*$ to make sure that in the construction of $\mathfrak{F}_{\beta+\beta'}$ defined by (3.15), the subalgebra $\epsilon(F_{\beta+\beta'})_+$ is abelian so everything there still works.

Chapter 4

Affine W-algebras associated to truncated current Lie algebras

In this chapter, we define classical and quantum affine W-algebras associated to truncated current Lie algebras.

4.1 Vertex algebras and Poisson vertex algebras

For a vector space V, the vector space of formal Laurant series with coefficients in V is defined to be

$$V[[z,z^{-1}]] = \left\{ \sum_{n\in\mathbb{Z}} v_n z^n \mid v_n \in V \right\}.$$

It contains the following subspaces

$$V[z] = \left\{ \sum_{n=0}^{N} v_n z^n \mid v_n \in V, N \in \mathbb{Z}_{\geq 0} \right\}, \qquad V[[z]] = \left\{ \sum_{n\geq 0} v_n z^n \mid v_n \in V \right\}$$

and

$$V((z)) = \left\{ \sum_{n\in\mathbb{Z}} v_n z^n \mid v_n \in V, v_n = 0 \text{ for } n \ll 0 \right\}.$$

When V is a vector superspace, an element $v(z) = \sum_n v_n z^n \in V[[z,z^{-1}]]$ is called homogeneous if all of the coefficients v_n have the same parity, which is also defined to be the parity of $v(z)$. The formal differential and the formal residue of $v(z)$ are defined respectively as follows,

$$\partial_z v(z) := \sum_{n\in\mathbb{Z}} n v_n z^{n-1}, \qquad \mathrm{Res}_z v(z) := v_{-1}.$$

Definition 4.1.1. A *vertex superalgebra* is a quadruple $(V, |0\rangle, Y, T)$, where V is a vector superspace, $|0\rangle \in V$ is an even element called the *vacuum vector*, $Y : V \to \mathrm{End}\, V[[z,z^{-1}]]$ is a parity-preserving map sending a to $Y(a,z) := \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$ called the *vertex operator* associated to a, and $T : V \to$

V is the map defined by $Ta = a_{-2}|0\rangle$, called the *infinitesimal translation operator*. These data are required to satisfy the following axioms for all $a, b \in V$,

(i) Truncation: $a_n b = 0$ for $n \gg \infty$,

(ii) Vacuum: $T|0\rangle = 0, Y(a,z)|0\rangle|_{z=0} = a$, i.e., $a_n|0\rangle = \delta_{n,-1}a$ for $n \geq -1$,

(iii) Translation covariance: $[T, Y(a,z)] = \partial_z Y(a,z)$, i.e., $[T, a_n] = -na_{n-1}$,

(iv) Locality: $(z-w)^{N(a,b)}[Y(a,z), Y(b,w)] = 0$ for some $N(a,b) \in \mathbb{Z}_{\geq 0}$.

We call V a *vertex algebra* when V is a purely even vector space.

Here we follow the definition of V. Kac [Kac98], while R. Borcherds [Bor86] originally used the *Jacobi identity* instead of the axiom of locality: for $\ell, m, n \in \mathbb{Z}$ and $u, v \in V$,

$$\sum_{i\geq 0}(-1)^i \binom{\ell}{i} \left(u_{m+\ell-i}v_{n+i} - (-1)^{\ell+p(u)p(v)} v_{n+\ell-i}u_{m+i}\right) = \sum_{i\geq 0} \binom{m}{i} (u_{\ell+i}v)_{m+n-i}. \tag{4.1}$$

The equivalence between the Jacobi identity and the axiom of locality can be found in [DSK06]. From the Jacobi identity (4.1), one can get the following useful formulas [LL04, Kac17],

$$\text{commutator formula:} \quad [u_m, v_n] = \sum_{i\geq 0} \binom{m}{i} (u_i v)_{m+n-i}, \tag{4.2}$$

$$\text{skew-symmetry:} \quad u_m v = (-1)^{p(u)p(v)} \sum_{i=0}^{m} (-1)^{m+i+1} \frac{T^i}{i!} v_{m+i}u, \tag{4.3}$$

$$\text{iterate formula:} \quad (u_m v)_n = \sum_{i\geq 0}(-1)^i \binom{m}{i} \left(u_{m-i}v_{n+i} - (-1)^{m+p(u)p(v)} v_{m+n-i}u_i\right). \tag{4.4}$$

A vertex superalgebra V is called commutative if $[a_m, b_n] = 0$ for all $a, b \in V$ and $m, n \in \mathbb{Z}$. It is known that V is commutative if and only if $a_n = 0$ for all $a \in V$ and $n \geq 0$. Moreover, if V is not commutative, then the number $N(a,b)$ in the axiom of locality is not bounded. Indeed, a commutative vertex superalgebra is the same thing as a unital commutative associative superalgebra with a derivation. See [FHL93, LL04, Kac98] for details.

Let V be vertex superalgebra. The λ-*bracket* of $a, b \in V$ is defined to be

$$[a_\lambda b] = \sum_{n\geq 0} \frac{\lambda^n}{n!} a_n b = \mathrm{Res}_z e^{\lambda z} Y(a,z)b.$$

By the truncation axiom, $[a_\lambda b] \in V[\lambda]$ is a polynomial in λ with coefficients in V. The λ-bracket satisfies the following properties [Kac17], for all $a, b, c \in V$,

$$\text{sesquilinearity:} \quad [(Ta)_\lambda b] = -\lambda[a_\lambda b], \quad [a_\lambda (Tb)] = (\lambda + T)[a_\lambda b], \tag{4.5}$$

$$\text{skew-symmetry:} \quad [b_\lambda a] = -(-1)^{p(a)p(b)}[a_{-\lambda-T} b], \tag{4.6}$$

$$\text{Jacobi identity:} \quad [a_\lambda [b_\mu c]] - (-1)^{p(a)p(b)}[b_\mu [a_\lambda c]] = [[a_\lambda b]_{\lambda+\mu} c]. \tag{4.7}$$

Definition 4.1.2. A *Lie conformal superalgebra* is a vector superspace R admitting a $\mathbb{C}[T]$-module structure, where T is an indeterminate that acts on R as an even endomorphism, endowed with a $\mathbb{C}$-bilinear λ-bracket $[\cdot_\lambda\cdot] : R \otimes R \to \mathbb{C}[\lambda] \otimes R$, such that (4.5), (4.6) and (4.7) are satisfied.

Let V be a vertex superalgebra and $a(z) = Y(a, z), b(z) = Y(b, z)$ for $a, b \in V$. The *normal ordered product* of $a(z)$ and $b(z)$ is defined to be

$$: a(z)b(z) := a(z)_+ b(z) + (-1)^{p(a)p(b)} b(z)a(z)_-,$$

where $a(z)_+ := \sum_{n<0} a_n z^{-n-1}$ and $a(z)_- := \sum_{n\geq 0} a_n z^{-n-1}$. One can show that $: a(z)b(z) := Y(a_{-1}b, z)$. The normal ordered product of several vertex operators is defined from right to left, and we have $: a(z)b(z)c(z) := Y(a_{-1}(b_{-1}c), z)$.

The coefficients of $Y(a, z)$ are called the *Fourier coefficients* or *modes* of a. The zero mode a_0 will play an important role in the sequel. The minus one mode a_{-1} is usually considered as a product in V. Indeed, $: ab := a_{-1}b$ defines an algebra structure on V, where the vacuum vector $|0\rangle$ plays the role of unit and T plays a role of derivation, i.e., $(V, |0\rangle, ::, T)$ is a unital differential superalgebra. With respect to $::$, V is usually neither commutative nor associative, but we have (see [Kac17])

$$\text{weak-commutativity:} \quad : ab : -(-1)^{p(a)p(b)} : ba := \sum_{n\geq 0} (-1)^n \frac{T^{n+1}}{(n+1)!} a_n b = \int_{-T}^{0} [a_\lambda b] d\lambda, \tag{4.8}$$

$$\begin{aligned} \text{weak-associativity:} \quad & :: ab : c : - : a : bc :: \\ &= \sum_{n\geq 0} \left(: \frac{T^{n+1}a}{(n+1)!} (b_n c) : +(-1)^{p(a)p(b)} : \frac{T^{n+1}b}{(n+1)!} (a_n c) : \right) \\ &=: \left(\int_0^T d\lambda a \right) [b_\lambda c] : +(-1)^{p(a)p(b)} : \left(\int_0^T d\lambda b \right) [a_\lambda c] : . \end{aligned} \tag{4.9}$$

Remark 4.1.3. *Given a polynomial $f(\lambda) = \sum_{i=0}^{n} \lambda^i v_i \in V[\lambda]$, where V is a vector space, we define*

$$\int_A^B f(\lambda) d\lambda = \sum_{i=0}^{n} \frac{B^{i+1} - A^{i+1}}{i+1} v_i.$$

When A, B are operators acting on V, we write $d\lambda$ before the element they act on if it is not clear. For example, in (4.9), T acts on a in the first term and on b in the second term.

The following is a λ-bracket version of the definition of vertex superalgebras [DSK06].

Definition 4.1.4. A vertex superalgebra is a quintuple $(V, |0\rangle, T, [\cdot_\lambda\cdot], ::)$, such that

(i) $(V, [\cdot_\lambda\cdot], T)$ is a Lie conformal superalgebra by considering V as a $\mathbb{C}[T]$-module,

(ii) $(V, |0\rangle, ::, T)$ is a unital differential superalgebra satisfying (4.8) and (4.9),

(iii) The product :: and the λ-bracket $[\cdot_\lambda\cdot]$ are related by the non-commutative Wick formula

$$[a_\lambda :bc:] =: [a_\lambda b]c: + (-1)^{p(a)p(b)} : b[a_\lambda c]: + \int_0^\lambda [[a_\lambda b]_\mu c]d\mu. \tag{4.10}$$

A Poisson algebra (Definition 1.1.1) is a commutative associative algebra with another Lie bracket such that the associative multiplication and the Lie bracket satisfy Leibniz's rule. The notion of a Poisson vertex superalgebra can be introduced in a similar way.

Definition 4.1.5 ([Li05]). A *vertex Lie superalgebra* is a triple (V, Y_-, D), where V is a vector superspace, D is a linear operator $D : V \to V$ and Y_- is a parity-preserving linear map

$$Y_- : V \to \mathrm{End}\, V[[z, z^{-1}]], \qquad v \mapsto Y_-(v, z) = \sum_{n \geq 0} v_{[n]} z^{-n-1},$$

such that $u_{[n]}v = 0$ for $n \gg 0$, $(Dv)_{[n]} = -nv_{[n-1]}$ for $u, v \in V$ and $n \geq 0$. Moreover, we require (4.2) and (4.3) to be satisfied for all $u, v \in V$ and $m, n \in \mathbb{Z}_{\geq 0}$ if we replace a_n by $a_{[n]}$, etc.

Remark 4.1.6. *Note that we use a_n for the Fourier coefficients of the linear map Y in a vertex superalgebra, and $a_{[n]}$ for the coefficients of the linear map Y_- in a vertex Lie superalgebra.*

Definition 4.1.7. A *Poisson vertex superalgebra* is a commutative vertex superalgebra $(V, |0\rangle, Y, T)$, with a vertex Lie superalgebra structure (V, Y_-, T) such that for all $a, b, c \in V$ and $n \geq 0$, we have

$$a_{[n]}(b_{-1}c) = (a_{[n]}b)_{-1}c + (-1)^{p(a)p(b)} b_{-1}(a_{[n]}c).$$

Notation: For a Poisson vertex superalgebra $(V, |0\rangle, Y, Y_-, T)$, since $a_n = 0$ for all $a \in V$ and $n \geq 0$, where a_n is the Fourier coefficients of the vertex operator $Y(a, z)$, we denote by $a_n = a_{[n]}$ for $n \geq 0$, where $a_{[n]}$ is the Fourier coefficients of $Y_-(a, z)$.

Note that in a Poisson vertex superalgebra V, (4.8) and (4.9) become

$$:ab: = (-1)^{p(a)p(b)} :ba: \qquad \text{and} \qquad ::ab:c: =: a:bc::,$$

so $(V, ::)$ is both commutative and associative. For $a, b \in V$, we denote by

$$\{a_\lambda b\} = \sum_{n \geq 0} a_n b. \tag{4.11}$$

One can show that $\{\cdot_\lambda\cdot\}$ also satisfies (4.5), (4.6) and (4.7), and we have the following equivalent definition of a Poisson vertex superalgebra [DSK06].

Definition 4.1.8. A Poisson vertex superalgebra is a quintuple $(V, |0\rangle, T, \{\cdot_\lambda\cdot\}, ::)$ such that

(i) $(V, |0\rangle, ::, T)$ is a unital associative and commutative differential superalgebra,

(ii) $(V, \{\cdot_\lambda\cdot\}, T)$ is a Lie conformal superalgebra,

(iii) The product :: and the λ-bracket $\{\cdot_\lambda\cdot\}$ are related by the commutative Wick formula

$$\{a_\lambda : bc :\} =: \{a_\lambda b\}c : +(-1)^{p(a)p(b)} : b\{a_\lambda c\} : . \tag{4.12}$$

Remark 4.1.9. *Compared to the λ-bracket definition of a vertex superalgebra, we do not have the integral terms in (4.8), (4.9) and (4.10) for a Poisson vertex superalgebra.*

Let V be a vertex superalgebra. The commutator formula (4.2) implies that

$$[a_0, b_n] = (a_0 b)_n. \tag{4.13}$$

Lemma 4.1.10. *Let $(V, |0\rangle, Y, T)$ be a vertex superalgebra and $d \in V$ satisfying $d_0^2 = 0$. Then the homology $H(V, d_0) := \dfrac{\ker\ d_0}{\operatorname{im} d_0}$ inherits a vertex superalgebra structure from that of V.*

Proof. Recall that $Y(d, z) = \sum_{n\in\mathbb{Z}} d_n z^{-n-1}$ and d_0 is the zero mode of d. It is enough to prove that for all $a \in \ker\ d_0$, $Y(a, z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$ is a well-defined element in $\operatorname{End} H(V, d_0)[[z, z^{-1}]]$, i.e., a_n preserves both $\ker\ d_0$ and $\operatorname{im} d_0$. From (4.13), for all $b \in V$, we have

$$[d_0, a_n]b = (d_0 a)_n b = 0, \quad \text{i.e.,} \quad d_0(a_n b) = (-1)^{p(d)p(a)} a_n(d_0 b). \tag{4.14}$$

Let $u \in \ker\ d_0$ and $v \in \operatorname{im} d_0$ with $v = d_0 w$. Then (4.14) implies that $a_n u \in \ker\ d_0$ and

$$a_n v = a_n(d_0 w) = (-1)^{p(d)p(a)} d_0(a_n w) \in \operatorname{im} d_0.$$

Therefore, $Y(a, z)$ is a well-defined element in $\operatorname{End} H(V, d_0)[[z, z^{-1}]]$, for all $a \in \ker\ d_0$. ☐

Remark 4.1.11. *$H(V, d_0)$ is a Poisson vertex superalgebra if V is a Poisson vertex superalgebra.*

4.2 Non-linear Lie conformal algebras and their universal enveloping vertex algebras

Definition 4.2.1. A *non-linear Lie superalgebra* is a vector superspace L, equipped with a parity-preserving linear map $[\cdot, \cdot]$, $L \otimes L \to L \oplus \mathbb{C}$, where $\mathbb{C}$ is defined to be even, such that for all $a, b, c \in L$, we have $[a, b] = -(-1)^{p(a)p(b)}[b, a]$ and the Jacobi identity holds:

$$[a, [b, c]] = [[a, b], c] + (-1)^{p(a)p(b)}[b, [a, c]]. \tag{4.15}$$

We assume that $[\mathbb{C}, L] = 0$ in the Jacobi identity. Note that $L \oplus \mathbb{C}$ is a Lie superalgebra.

Definition 4.2.2. A *non-linear Lie conformal superalgebra* is a $\mathbb{C}[T]$-module R, endowed with a λ-bracket $[\cdot_\lambda\cdot] : R \otimes R \to \mathbb{C}[\lambda] \otimes (R \oplus \mathbb{C})$, such that (4.5), (4.6) and (4.7) are satisfied.

Remark 4.2.3. *In the definition of a non-linear Lie conformal superalgebra, we can understand that $T\mathbb{C} = 0$ so that $R \oplus \mathbb{C}$ is a Lie conformal superalgebra. We should also understand that $[\mathbb{C}_\lambda R] = 0$ in the Jacobi identity. There is a more general version of a non-linear Lie conformal algebra in [DSK06].*

Given a non-linear Lie conformal superalgebra R, define a bracket $[\cdot,\cdot] : R \otimes R \to R \oplus \mathbb{C}$ by

$$[a,b] := -\sum_{n\geq 0} \frac{(-T)^{n+1}}{(n+1)!} a_n b = \int_{-T}^{0} [a_\lambda b] d\lambda. \tag{4.16}$$

Lemma 4.2.4 ([BK03]). *The Lie bracket (4.16) defines a non-linear Lie superalgebra structure on R.*

Proof. The bracket is obviously bilinear and takes values in $R \oplus \mathbb{C}$. We only need to verify skew-symmetry and the Jacobi identity. We have

$$\int_{-T}^{0} [a_\lambda b] d\lambda = -(-1)^{p(a)p(b)} \int_{-T}^{0} [b_{-\lambda-T} a] d\lambda = (-1)^{p(a)p(b)} \int_{0}^{-T} [b_\mu a] d\mu,$$

i.e., $[a,b] = -(-1)^{p(a)p(b)}[b,a]$.

For the Jacobi identity, note that

$$[a,[b,c]] = \int_{-T}^{0} \left[a_\lambda \int_{-T}^{0} [b_\mu c] d\mu \right] d\lambda = \int_{-T}^{0} \int_{-\lambda-T}^{0} [a_\lambda [b_\mu c]] d\mu d\lambda.$$

Similarly, we have

$$\begin{aligned}
[[a,b],c] &= \int_{-T}^{0} \left[\int_{-T}^{0} [a_\lambda b] d\lambda_\mu c \right] d\mu \\
&= \int_{-T}^{0} \int_{\mu}^{0} [[a_\lambda b]_\mu c]] d\lambda d\mu \\
&= \int_{-T}^{0} \int_{-T}^{\lambda} [[a_\lambda b]_\mu c]] d\mu d\lambda \\
&= \int_{-T}^{0} \int_{-\lambda-T}^{0} [[a_\lambda b]_{\mu'+\lambda} c]] d\mu' d\lambda,
\end{aligned}$$

and

$$\begin{aligned}
[b,[a,c]] &= (-1)^{p(a)p(b)} \int_{-T}^{0} \left[b_\mu \int_{-T}^{0} [a_\lambda c] d\lambda \right] d\mu \\
&= (-1)^{p(a)p(b)} \int_{-T}^{0} \int_{-\mu-T}^{0} [b_\mu [a_\lambda c]] d\lambda d\mu \\
&= (-1)^{p(a)p(b)} \int_{-T}^{0} \int_{-\lambda-T}^{0} [b_\mu [a_\lambda c]] d\mu d\lambda.
\end{aligned}$$

Now the Jacobi identity (4.15) comes from the Jacobi identity (4.7). □

The non-linear Lie superalgebra structure on R defined by (4.16) is denoted by R_{Lie}.

Remark 4.2.5. *Let $(R, [\cdot_\lambda\cdot], T)$ be a non-linear Lie conformal superalgebra. Another construction of R_{Lie} is as follows [Kac98]. Let $\tilde{R} = (R \oplus \mathbb{C}) \otimes \mathbb{C}[t,t^{-1}]$, where t and t^{-1} are considered as even elements. Let $\tilde{T} = T \otimes 1 + 1 \otimes \partial_t$. Define*

$$(a \otimes f)_n (b \otimes g) = \sum_{i\geq 0} (a_{n+i} b) \otimes \left(\frac{\partial_t^i f}{i!} g \right)$$

and

$$[(a\otimes f)_\lambda (b\otimes g)] = \sum_{n\geq 0} \frac{\lambda^n}{n!}(a\otimes f)_n(b\otimes g).$$

Then one can show [Kac98] that $\tilde{R}$ is a Lie conformal superalgebra and $Lie\, R = \tilde{R}/\tilde{T}\tilde{R}$ is a Lie superalgebra with Lie bracket

$$[a\otimes t^m, b\otimes t^n] = \sum_{i\geq 0}(a_i b)\otimes t^{m+n-i}.$$

Assume that $R = (\mathbb{C}[T]\otimes U)\oplus S$, where $\mathbb{C}[T]\otimes U$ is the free part and S the torsion part of R as a $\mathbb{C}[T]$-module. Then it can be proved that

$$Lie\, R \cong U\otimes [t,t^{-1}]\oplus S\otimes t^{-1}\oplus \mathbb{C}\otimes t^{-1} \tag{4.17}$$

as vector spaces. Let $(Lie\, R)_-$ be the $\mathbb{C}$-span of the images of $a\otimes t^n$ for $a\in R$ and $n\geq 0$, and $(Lie\, R)_+$ be the $\mathbb{C}$-span of the images of $a\otimes t^n$ for $a\in R$ and $n<0$. Then both $(Lie\, R)_\pm$ are non-linear Lie subalgebras of $Lie\, R$ (by identifying $\mathbb{C}t^{-1}$ with $\mathbb{C}$). The subalgebra $(Lie\, R)_-$ is called the annihilation algebra of R and it plays important roles in the representation theory of R. The subalgebra $(Lie\, R)_+$ is isomorphic to R_{Lie} by sending $s\otimes t^{-1}$ to s for $s\in S$ and $u\otimes t^{-n}$ to $\dfrac{(-T)^{n-1}u}{(n-1)!}$ for $u\in U$ and $n\geq 1$.

Let R be a non-linear Lie conformal superalgebra, and $S(R)$ the symmetric algebra of R. Then $S(R)$ is naturally a commutative superalgebra. The action of T on R can be extended to $S(R)$ by requiring $T(ab) = T(a)b + aT(b)$ for all $a,b\in S(R)$. Therefore, $S(R)$ is a unital commutative differential associative superalgebra hence a commutative vertex superalgebra. Define a λ-bracket on $S(R)$ by letting $\{\cdot_\lambda\cdot\} = [\cdot_\lambda\cdot] : R\times R\to S(R)$ be the λ-bracket of R, and then extend it to $S(R)\times S(R)$ by requiring (4.12). Then one can show that these data define a Poisson vertex superalgebra on $S(R)$.

Proposition 4.2.6. *Let R be a non-linear Lie conformal superalgebra, and $S(R)$ the symmetric algebra of R. Then there is a Poisson vertex superalgebra structure on $S(R)$, such that $\{\cdot_\lambda\cdot\} : R\times R\to S(R)$ is the λ-bracket of R.*

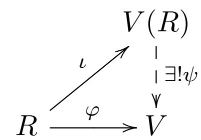
Let R_{Lie} be the non-linear Lie superalgebra defined by (4.16). The universal enveloping algebra of R_{Lie} is defined to be $U(R_{Lie}) = T(R_{Lie})/I$, where $T(R_{Lie})$ is the tensor algebra of R_{Lie} and I is the two-sided ideal of $T(R_{Lie})$ generated by $a\otimes b - (-1)^{p(a)p(b)}b\otimes a - [a,b]$ for all $a,b\in R$.

Proposition 4.2.7 ([BK03])**.** *Let R be a non-linear Lie conformal superalgebra. Then the universal enveloping algebra $U(R_{Lie})$ of R_{Lie} has a vertex superalgebra structure, where the λ-bracket on $R_{Lie}\times R_{Lie}$ is the λ-bracket of R, and the product :: on $R_{Lie}\times U(R_{Lie})$ is the product of $U(R_{Lie})$.*

Proof. The unit element of $U(R_{Lie})$ plays the role of the vacuum vector. The λ-bracket and the product :: can be extended to $U(R_{Lie})$ by (4.9) and (4.10) in a unique way. □

Definition 4.2.8. The vertex superalgebra $U(R_{Lie})$ is called the *universal enveloping vertex superalgebra* of R, and is usually denoted by $V(R)$.

Remark 4.2.9. *Like the universal enveloping algebra of a Lie algebra, $V(R)$ has the following universal property: (1) The natural inclusion $\iota : R \oplus \mathbb{C} \hookrightarrow V(R)$, while $\mathbb{C} \to \mathbb{C}|0\rangle$, is a Lie conformal superalgebra homomorphism; (2) Let V be a vertex superalgebra and $\varphi : R \oplus \mathbb{C} \to V$ a Lie conformal superalgebra homomorphism with $\mathbb{C} \to \mathbb{C}|0\rangle$. Then there exists a unique vertex superalgebra homomophism $\psi : V(R) \to V$ such that $\psi \circ \iota = \varphi$.*



Let L be a non-linear Lie superalgebra with an invariant supersymmetric bilinear form $(\cdot \mid \cdot)$. Let $k \in \mathbb{C}$ and $\mathrm{Cur}^k L := \mathbb{C}[T] \otimes L$. Set

$$[a_\lambda b] := [a, b] + \lambda k(a \mid b) \quad \text{for } a, b \in L. \tag{4.18}$$

Lemma 4.2.10. *There is a unique non-linear Lie conformal superalgebra structure on $\mathrm{Cur}^k L$ satisfying (4.18).*

Proof. Once the λ-bracket is well-defined for all $a, b \in L$, it extends uniquely to a λ-bracket on $\mathrm{Cur}^k L$ by (4.5). Skew-symmetry of $[\cdot_\lambda \cdot]$ comes from the skew-symmetry of the Lie bracket $[\cdot, \cdot]$ and the supersymmetry of $(\cdot \mid \cdot)$. The Jacobi identity of $[\cdot_\lambda \cdot]$ comes from the Jacobi identity of $[\cdot, \cdot]$. □

Let A be a finite-dimensional vector superspace and $\langle \cdot, \cdot \rangle$ a non-degenerate skew-supersymmetric bilinear form on A. Let $R(A) := \mathbb{C}[T] \otimes A$ and define

$$[a_\lambda b] := \langle a, b \rangle \quad \text{for } a, b \in A. \tag{4.19}$$

Lemma 4.2.11. *There is a unique non-linear Lie conformal superalgebra structure on $R(A)$ satisfying (4.19).*

Proof. Once the λ-bracket for all $a, b \in A$ are well-defined, it extends uniquely to a λ-bracket on $R(A)$ by (4.5). Skew-symmetry of $[\cdot_\lambda \cdot]$ comes from the skew-symmetry of $\langle \cdot, \cdot \rangle$. The Jacobi identity is trivial since we assume that $[\mathbb{C}_\lambda R] = 0$ in all non-linear Lie conformal superalgebras. □

Now let us consider the universal enveloping vertex superalgebras of $\mathrm{Cur}^k L$ and $R(A)$.

Example 4.2.12. Let $\mathfrak{g}$ be a Lie algebra with a non-degenerate invariant symmetric bilinear form $(\cdot \mid \cdot)$. The *Kac-Moody affinization* of $\mathfrak{g}$ is the Lie algebra

$$\hat{\mathfrak{g}} = \left(\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]\right) \oplus \mathbb{C}K$$

with Lie bracket:

$$[at^m, bt^n] = [a, b]t^{m+n} + m\delta_{m,-n}(a \mid b)K, \quad [K, \hat{\mathfrak{g}}] = 0.$$

Let $\hat{\mathfrak{g}}_+ = (\mathfrak{g} \otimes \mathbb{C}[t]) \oplus \mathbb{C}K$, which is a subalgebra of $\hat{\mathfrak{g}}$. Define a one-dimensional module $\mathbb{C}_k$ of $\hat{\mathfrak{g}}_+$ on which $\mathfrak{g} \otimes \mathbb{C}[t]$ acts as zero and K acts as the constant k. The *level k vacuum representation* of $\tilde{\mathfrak{g}}$ is the induced module

$$V^k(\mathfrak{g}) := \operatorname{Ind}_{\hat{\mathfrak{g}}_+}^{\hat{\mathfrak{g}}} \mathbb{C}_k.$$

It is isomorphic to $U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}])$ as a vector space. There is a unique vertex algebra structure on $V^k(\mathfrak{g})$ with the vacuum vector being the identity 1 and $a(z) := Y(at^{-1}, z) = \sum_{n\in\mathbb{Z}}(at^n)z^{-n-1}$ for all $a \in \mathfrak{g}$. It is called the *universal affine vertex algebra* of level k associated to $\mathfrak{g}$.

Remark 4.2.13. *Note that $Lie\,\mathrm{Cur}^k\,\mathfrak{g} \cong \hat{\mathfrak{g}}$ by considering $k \otimes t^{-1}$ as K, where $k \otimes t^{-1}$ is defined by (4.17). We have $(Lie\,\mathrm{Cur}^k\,\mathfrak{g})_- \cong \hat{\mathfrak{g}}_+$ and $(Lie\,\mathrm{Cur}^k\,\mathfrak{g})_+ \cong \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]$. Since $(\mathrm{Cur}^k\,\mathfrak{g})_{Lie} \cong (Lie\,\mathrm{Cur}^k\,\mathfrak{g})_+$, we have $V^k(\mathfrak{g}) \cong U((\mathrm{Cur}^k\,\mathfrak{g})_{Lie})$ as vector spaces. Comparing λ-brackets of $V^k(\mathfrak{g})$ and $U((\mathrm{Cur}^k\,\mathfrak{g})_{Lie})$, one can see that they are isomorphic as vertex algebras.*

Example 4.2.14. Let A be a finite-dimensional vector superspace and $\langle\cdot,\cdot\rangle$ be a non-degenerate skew-supersymmetric bilinear form on A. The *Clifford affinization* of A is the Lie superalgebra $\widehat{A} := (A \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}K$ with Lie bracket:

$$[at^m, bt^n] = \langle a, b\rangle\delta_{m,-n-1}K, \qquad [K, \widehat{A}] = 0.$$

Let $\widehat{A}_+ = (A \otimes \mathbb{C}[t]) \oplus \mathbb{C}K$, which is an abelian subalgebra of $\widehat{A}$. Let $\mathbb{C}$ be the one-dimensional representation of $\widehat{A}_+$, on which $A \otimes \mathbb{C}[t]$ acts as zero and K acts as the identity. The *Fock representation* of $\widehat{A}$ is the induced module

$$F(A) := \operatorname{Ind}_{\widehat{A}_+}^{\widehat{A}} \mathbb{C}.$$

It is isomorphic to $U(A\otimes t^{-1}\mathbb{C}[t^{-1}])$ as a vector space. There is a unique vertex superalgebra structure on $F(A)$ with the vacuum vector being the identity 1 and $a(z) := Y(at^{-1}, z) = \sum_{n\in\mathbb{Z}}(at^n)z^{-n-1}$ for $a \in A$. It is called the *vertex superalgebra of fermions* associated to A and $\langle\cdot,\cdot\rangle$.

Remark 4.2.15. *As in Remark 4.2.13, we have $F(A) \cong U(R(A)_{Lie})$. Indeed, we have $Lie\,R(A) \cong \widehat{A}$ by considering $1\otimes t^{-1}$ as K, where $1\otimes t^{-1}$ is defined in (4.17). Moreover, we have $(Lie\,R(A))_- \cong \widehat{A}_+$ and $(Lie\,R(A))_+ \cong A \otimes t^{-1}\mathbb{C}[t^{-1}]$. Since $R(A)_{Lie} \cong (Lie\,R(A))_+$, we have $F(A) \cong U(R(A)_{Lie})$ as vector spaces. Comparing their λ-brackets shows that they are isomorphic as vertex superalgebras.*

Here are two examples.

Example 4.2.16. Let A^{ne} be a finite-dimensional vector space and $\langle\cdot,\cdot\rangle$ be a non-degenerate skew-symmetric bilinear form on A^{ne}. By Lemma 4.2.11, we have a non-linear Lie conformal algebra $R(A^{ne})$. By Example 4.2.14, we have its universal enveloping vertex algebra $F(A^{ne})$, which is called the *vertex algebra of neutral fermions* associated to A^{ne} and $\langle\cdot,\cdot\rangle$.

Example 4.2.17. Let V be a finite-dimensional vector space and V^* be its dual. Let $\iota(V)$ and $\varepsilon(V^*)$ be copies of V and V^*, respectively, but considered as purely odd spaces. For $v \in V$ and $u^* \in V^*$, let $\iota(v)$ and $\varepsilon(u^*)$ be the corresponding elements in $\iota(V)$ and $\varepsilon(V^*)$. Let $A^{ch} = \iota(V) \oplus \varepsilon(V^*)$, and endow A^{ch} with a non-degenerate skew-supersymmetric bilinear form $\langle \cdot, \cdot \rangle$ by setting $\langle \iota(V), \iota(V) \rangle = \langle \varepsilon(V^*), \varepsilon(V^*) \rangle = 0$ and $\langle \iota(v), \varepsilon(u^*) \rangle = \langle \varepsilon(u^*), \iota(v) \rangle = u^*(v)$ for all $v \in V, u^* \in V^*$. By Lemma 4.2.11, we have a non-linear Lie conformal superalgebra $R(A^{ch})$. By Example 4.2.14, we have its universal enveloping vertex superalgebra $F(A^{ch})$, which is called the *vertex superalgebra of charged superfermions* associated to A^{ch}.

4.3 Affine W-algebras associated to truncated current Lie algebras

Now let us come back to the basic setting of Chapter 2. Let $\mathfrak{g}$ be a finite-dimensional semi-simple Lie algebra over $\mathbb{C}$ with a non-degenerate invariant symmetric bilinear form $(\cdot \mid \cdot)$. Let $\mathfrak{g}_p$ be the level p truncated current Lie algebra associated to $\mathfrak{g}$ and $(\cdot \mid \cdot)_p$ a fixed non-degenerate invariant bilinear form on $\mathfrak{g}_p$. Let $\Gamma : \mathfrak{g} \overset{\operatorname{ad} h_\Gamma}{=\!=} \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i)$ be a good $\mathbb{Z}$-grading of $\mathfrak{g}$ with a good element $e \in \mathfrak{g}(2)$, and $\{e, f, h\}$ an $s\ell_2$-triple with $h \in \mathfrak{g}(0)$ and $f \in \mathfrak{g}(-2)$. Let

$$\mathfrak{g}_p = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_p(i), \ \text{ where } \mathfrak{g}_p(i) := \{y \in \mathfrak{g}_p \mid [h_\Gamma, y] = iy\} = \mathfrak{g}(i)_p \tag{4.20}$$

be the corresponding good $\mathbb{Z}$-grading of $\mathfrak{g}_p$, with the same good element e. By Lemma 2.2.5, the bilinear form $\langle \cdot, \cdot \rangle_p$ on $\mathfrak{g}_p(-1)$ defined by $\langle a, b \rangle_p = (e \mid [a, b])_p$ for $a, b \in \mathfrak{g}_p(-1)$ is non-degenerate.

Let $A_p^{ne} = \epsilon(\mathfrak{g}_p(-1))$ be a copy of $\mathfrak{g}_p(-1)$. For $x \in \mathfrak{g}_p(-1)$, let $\epsilon(x)$ be the corresponding element of A_p^{ne}. More generally, for $x \in \mathfrak{g}_p$, we write $\epsilon(x) = \epsilon(x_{-1})$, where $x = \sum_i x_i$ and $x_i \in \mathfrak{g}_p(i)$. The form $\langle \epsilon(x), \epsilon(y) \rangle_p := \langle x, y \rangle_p$ is skew-symmetric and non-degenerate on A_p^{ne}. By Lemma 4.2.11 and Example 4.2.16, we have a non-linear Lie conformal algebra $R(A_p^{ne})$ and its universal enveloping vertex algebra $F(A_p^{ne})$.

Let $\mathfrak{n}_p = \bigoplus_{j<0} \mathfrak{g}_p(j)$ and $\mathfrak{n}_p^*$ be the dual of $\mathfrak{n}_p$. Let $\iota(\mathfrak{n}_p)$ and $\varepsilon(\mathfrak{n}_p^*)$ be copies of $\mathfrak{n}_p$ and $\mathfrak{n}_p^*$, respectively, but considered as purely odd spaces. For $u \in \mathfrak{n}_p$ and $v^* \in \mathfrak{n}_p^*$, let $\iota(u)$ and $\varepsilon(v^*)$ be the corresponding elements of $\iota(\mathfrak{n}_p)$ and $\varepsilon(\mathfrak{n}_p^*)$, respectively. Let $A_p^{ch} = \iota(\mathfrak{n}_p) \oplus \varepsilon(\mathfrak{n}_p^*)$. By Lemma 4.2.11 and Example 4.2.17 we have a non-linear Lie conformal superalgebra $R(A_p^{ch})$ and its universal enveloping vertex superalgebra $F(A_p^{ch})$.

For $\mathfrak{g}_p$ and the bilinear form $(\cdot \mid \cdot)_p$, we have the non-linear Lie conformal algebra $\operatorname{Cur}^k \mathfrak{g}_p$ by Lemma 4.2.10 and its universal enveloping vertex algebra $V^k(\mathfrak{g}_p)$ by Example 4.2.12.

Let us choose a basis $\{u_\alpha\}_{\alpha\in S_p^j}$ of $\mathfrak{g}_p(j)$ for each j. Let $S_p^- = \bigcup_{j<0} S_p^j$ and $S_p' = S_p^{-1}$. Then $\{u_\alpha\}_{\alpha\in S_p^-}$ forms a basis of $\mathfrak{n}_p$ and $\{\epsilon(u_\alpha)\}_{\alpha\in S_p'}$ forms a basis of A_p^{ne}. Let $\{u_\alpha^*\}_{\alpha\in S_p^-}$ be the dual basis of $\mathfrak{n}_p^*$, with $\langle u_\alpha^*, u_\beta \rangle = \delta_{\alpha,\beta}$. Then $\{\iota(u_\alpha)\}_{\alpha\in S_p^-}$ and $\{\varepsilon(u_\alpha^*)\}_{\alpha\in S_p^-}$ form dual bases of $\iota(\mathfrak{n}_p)$ and $\varepsilon(\mathfrak{n}_p^*)$, respectively. Let $\{c_{i,j}^k\}$ be the structure constants of $\mathfrak{g}_p$ with respect to the basis $\{u_i\}$, i.e., $[u_i, u_j] = \sum_k c_{i,j}^k u_k$.

4.3.1 Classical affine W-algebras through classical Drinfeld-Sokolov reduction

Let

$$R^k(\mathfrak{g}_p, e) = \mathrm{Cur}^k\, \mathfrak{g}_p \oplus R(A_p^{ch}) \oplus R(A_p^{ne}) = \mathbb{C}[T] \otimes \Big(\mathfrak{g}_p \oplus A_p^{ch} \oplus A_p^{ne}\Big)$$

be the direct sum of three non-linear Lie conformal superalgebras and $S(R^k(\mathfrak{g}_p, e))$ its symmetric algebra. By Proposition 4.2.6, $S(R^k(\mathfrak{g}_p, e))$ has a Poisson vertex superalgebra structure.

Let

$$\bar{d}^p = \sum_{i \in S_p^-} \varepsilon(u_i^*)\, (u_i + (e \mid u_i)_p + \epsilon(u_i)) - \frac{1}{2} \sum_{i,j \in S_p^-} \iota([u_i, u_j])\varepsilon(u_i^*)\varepsilon(u_j^*).$$

Obviously, $\bar{d}^p \in S(R^k(\mathfrak{g}_p, e))$ is an odd element.

Lemma 4.3.1. *We have the following formulas for the λ-bracket of $\bar{d}^p$ in $S(R^k(\mathfrak{g}_p, e))$,*

(1) $\{\bar{d}^p{}_\lambda x\} = \sum_{i \in S_p^-} ([u_i, x] + k(x \mid u_i)_p(\lambda + T))\varepsilon(u_i^*)$ *for* $x \in \mathfrak{g}_p$*;*

(2) $\{\bar{d}^p{}_\lambda \epsilon(y)\} = \sum_{i \in S_p^-} (e \mid [u_i, y])_p \varepsilon(u_i^*)$ *for* $y \in \mathfrak{g}_p(-1)$*;*

(3) $\{\bar{d}^p{}_\lambda \varepsilon(v^*)\} = -\frac{1}{2} \sum_{i,j \in S_p^-} \langle v^*, [u_i, u_j]\rangle \varepsilon(u_i^*)\varepsilon(u_j^*)$ *for* $v^* \in \mathfrak{n}_p^*$*;*

(4) $\{\bar{d}^p{}_\lambda \iota(u)\} = \sum_{i,j \in S_p^-} \langle u_i^*, u\rangle \Big(\iota([u_i, u_j])\varepsilon(u_j^*) + u_i + (e \mid u_i)_p + \epsilon(u_i)\Big)$ *for* $u \in \mathfrak{n}_p$*.*

Proof. Let $X(u_i) = u_i + (e \mid u_i)_p + \epsilon(u_i)$ and write $\bar{d}^p = \bar{d}^{p,1} + \bar{d}^{p,2}$, where

$$\bar{d}^{p,1} = \sum_{i \in S_p^-} \varepsilon(u_i^*) X(u_i) \quad \text{and} \quad \bar{d}^{p,2} = -\frac{1}{2} \sum_{i,j \in S_p^-} \iota([u_i, u_j])\varepsilon(u_i^*)\varepsilon(u_j^*).$$

Instead of calculating $\{\bar{d}^p{}_\lambda \cdot\}$, we will calculate $\{\cdot_\lambda \bar{d}^p\}$ and then use skew-symmetry to get the formulas for $\{\bar{d}^p{}_\lambda \cdot\}$. Our calculations are based on (4.12).

We have $\{a_\lambda b\} = 0$ if a, b come from different summands of $R^k(\mathfrak{g}_p, e)$. In particular, $\{x_\lambda \bar{d}^{p,2}\} = \{\epsilon(y)_\lambda \bar{d}^{p,2}\} = 0$ for all $x \in \mathfrak{g}_p$ and $y \in \mathfrak{g}_p(-1)$. By (4.12), we have

$$\{x_\lambda \bar{d}^p\} = \{x_\lambda \bar{d}^{p,1}\} = \sum_{i \in S_p^-} \varepsilon(u_i^*)\{x_\lambda X(u_i)\} = \sum_{i \in S_p^-} \varepsilon(u_i^*)([x, u_i] + \lambda k(x \mid u_i)_p) \tag{4.21}$$

and

$$\{\epsilon(y)_\lambda \bar{d}^p\} = \{\epsilon(y)_\lambda \bar{d}^{p,1}\} = \sum_{i \in S_p^-} \varepsilon(u_i^*)\{\epsilon(y)_\lambda X(u_i)\} = \sum_{i \in S_p^-} (e \mid [y, u_i])_p \varepsilon(u_i^*). \tag{4.22}$$

We obviously have $\{\varepsilon(v^*)_\lambda \bar{d}^{p,1}\} = 0$, so

$$\{\varepsilon(v^*)_\lambda \bar{d}^p\} = -\frac{1}{2} \sum_{i,j \in S_p^-} \{\varepsilon(v^*)_\lambda \iota([u_i, u_j])\}\varepsilon(u_i^*)\varepsilon(u_j^*) = -\frac{1}{2} \sum_{i,j \in S_p^-} \langle v^*, [u_i, u_j]\rangle \varepsilon(u_i^*)\varepsilon(u_j^*). \tag{4.23}$$

Finally, since $\{\iota(u)_\lambda X(u_i)\} = 0$, we have

$$\{\iota(u)_\lambda \bar{d}^{p,1}\} = \sum_{i\in S_p^-} \{\iota(u)_\lambda \varepsilon(u_i^*)\} X(u_i) = \sum_{i\in S_p^-} \langle u_i^*, u\rangle X(u_i), \tag{4.24}$$

and

$$\begin{aligned}\{\iota(u)_\lambda \bar{d}^{p,2}\} &= \frac{1}{2}\sum_{i,j\in S_p^-} \iota([u_i,u_j])\langle u_i^*, u\rangle \varepsilon(u_j^*) - \frac{1}{2}\sum_{i,j\in S_p^-} \iota([u_i,u_j])\varepsilon(u_i^*)\langle u_j^*, u\rangle \\ &= \sum_{i,j\in S_p^-} \langle u_i^*, u\rangle \iota([u_i,u_j])\varepsilon(u_j^*). \end{aligned}\tag{4.25}$$

Applying skew-symmetry to (4.21), (4.22), (4.23) and (4.24)+(4.25), we get the desired formulas for $\{\bar{d}^p{}_\lambda \cdot\}$ in $S(R^k(\mathfrak{g}_p, e))$. □

Before stating the following proposition, let us recall that in a (Poisson) vertex superalgebra, if v is an odd element and satisfies $\{v_\lambda v\} = 0$, then $(v_0)^2 = 0$. Indeed, $\{v_\lambda v\} = 0$ implies that $v_0 v = 0$. But $2(v_0)^2 = v_0v_0 + v_0v_0 = [v_0, v_0] = (v_0 v)_0 = 0$ by (4.13).

Proposition 4.3.2. *We have* $\{\bar{d}^p{}_\lambda \bar{d}^p\} = 0$ *hence* $(\bar{d}^p_0)^2 = 0$.

Proof. Recall that $X(u_i) = u_i + (e \mid u_i)_p + \epsilon(u_i)$ and $\{c_{i,j}^k\}$ are the structure constants of $\mathfrak{g}_p$ with respect to the basis $\{u_i\}$. Using (4.12) and the formulas in Lemma 4.3.1, we have

$$\begin{aligned}\{\bar{d}^p{}_\lambda \bar{d}^{p,1}\} &= \sum_{\ell\in S_p^-} \{\bar{d}^p{}_\lambda \varepsilon(u_\ell^*)\} X(u_\ell) - \sum_{i\in S_p^-} \varepsilon(u_i^*)\{\bar{d}^p{}_\lambda X(u_i)\} \\ &= -\frac{1}{2}\sum_{i,j,\ell\in S_p^-} \langle u_\ell^*, [u_i,u_j]\rangle \varepsilon(u_i^*)\varepsilon(u_j^*) X(u_\ell) - \sum_{i,j\in S_p^-} \varepsilon(u_i^*)([u_j,u_i] + \lambda k(u_i \mid u_j)_p)\varepsilon(u_j^*) \\ &\quad - \sum_{i,j\in S_p^-} \varepsilon(u_i^*)(e \mid [u_j,u_i]_p)\varepsilon(u_j^*) \\ &= -\frac{1}{2}\sum_{i,j,\ell\in S_p^-} c_{i,j}^\ell \varepsilon(u_i^*)\varepsilon(u_j^*) X(u_\ell) + \sum_{i,j,\ell\in S_p^-} c_{i,j}^\ell \varepsilon(u_i^*)\varepsilon(u_j^*)(u_\ell + (e \mid u_\ell)_p) \\ &= \frac{1}{2}\sum_{i,j,\ell\in S_p^-} c_{i,j}^\ell \varepsilon(u_i^*)\varepsilon(u_j^*) X(u_\ell). \end{aligned}$$

In the above calculations, we used the fact that $(u_i \mid u_j)_p = 0$ for $i,j \in S_p^-$. Moreover, when $c_{i,j}^\ell \neq 0$ and $i,j \in S_p^-$, we have $u_\ell \in \bigoplus_{i\le -2} \mathfrak{g}_p(i)$, hence $\epsilon(u_\ell) = 0$ and $X(u_\ell) = u_\ell + (e \mid u_\ell)_p$.

For the other part, we have $\{\bar{d}^p{}_\lambda \bar{d}^{p,2}\} = A + B + C$, where

$$A = -\frac{1}{2} \sum_{i,j \in S_p^-} \{\bar{d}^p_\lambda \iota([u_i, u_j])\} \varepsilon(u_i^*) \varepsilon(u_j^*),$$

$$B = \frac{1}{2} \sum_{i,j \in S_p^-} \iota([u_i, u_j]) \{\bar{d}^p_\lambda \varepsilon(u_i^*)\} \varepsilon(u_j^*),$$

$$C = -\frac{1}{2} \sum_{i,j \in S_p^-} \iota([u_i, u_j]) \varepsilon(u_i^*) \{\bar{d}^p_\lambda \varepsilon(u_j^*)\}.$$

By the formulas in Lemma 4.3.1, we have

$$\begin{aligned} A &= -\frac{1}{2} \sum_{i,j,s,t \in S_p^-} \langle u_s^*, [u_i, u_j] \rangle \left(\iota([u_s, u_t]) \varepsilon(u_t^*) + X([u_i, u_j]) \right) \varepsilon(u_i^*) \varepsilon(u_j^*) \\ &= -\frac{1}{2} \sum_{i,j,s,t,\ell \in S_p^-} c_{i,j}^s \left(c_{s,t}^\ell \iota(u_\ell) \varepsilon(u_t^*) + X(u_s) \right) \varepsilon(u_i^*) \varepsilon(u_j^*) \\ &= -\frac{1}{2} \sum_{i,j,s,t,\ell \in S_p^-} c_{i,j}^s c_{s,t}^\ell \iota(u_\ell) \varepsilon(u_t^*) \varepsilon(u_i^*) \varepsilon(u_j^*) - \frac{1}{2} \sum_{i,j,s \in S_p^-} c_{i,j}^s X(u_s) \varepsilon(u_i^*) \varepsilon(u_j^*). \end{aligned}$$

By the formulas in Lemma 4.3.1, $\varepsilon(u_i^*)$ and $\{\bar{d}^p{}_\lambda \varepsilon(u_i^*)\}$ commute with each other, so

$$\begin{aligned} B + C &= \sum_{i,j \in S_p^-} \iota([u_i, u_j]) \{\bar{d}^p_\lambda \varepsilon(u_i^*)\} \varepsilon(u_j^*) \\ &= -\frac{1}{2} \sum_{i,j,s,t \in S_p^-} \iota([u_i, u_j]) \langle u_i^*, [u_s, u_t] \rangle \varepsilon(u_s^*) \varepsilon(u_t^*) \varepsilon(u_j^*) \\ &= \frac{1}{2} \sum_{i,j,s,t,\ell \in S_p^-} c_{i,j}^\ell c_{s,t}^i \iota(u_\ell) \varepsilon(u_s^*) \varepsilon(u_t^*) \varepsilon(u_j^*). \end{aligned}$$

Now it is clear that

$$A + B + C = -\frac{1}{2} \sum_{i,j,s \in S_p^-} c_{i,j}^s X(u_s) \varepsilon(u_i^*) \varepsilon(u_j^*).$$

Hence we have $\{\bar{d}^p{}_\lambda \bar{d}^p\} = 0$ and $(\bar{d}_0^p)^2 = 0$.

□

Definition 4.3.3. The *classical affine W-algebra* $\mathcal{W}^k(\mathfrak{g}_p, e)$ associated to the data $(\mathfrak{g}_p, e, k)$ is defined to be the homology $H(S(R^k(\mathfrak{g}_p, e)), \bar{d}_0^p)$. It inherits a Poisson vertex superalgebra structure from that of $S(R^k(\mathfrak{g}_p, e))$.

Remark 4.3.4. *Classical affine W-algebras ($p = 0$ and e regular case) were first discovered by Drinfeld and Sokolov [DS84]. They were used to construct hierarchies on some infinite-dimensional Poisson manifolds. P. Casati [Cas11] generalized Drinfeld and Sokolov's method and constructed hierarchies on affinizations of truncated current Lie algebras. The authors of [DSKV13] constructed more* *general integrable hierarchies corresponding to other classical affine W-algebras ($p = 0$ case). The classicial affine W-algebra $\mathcal{W}^k(\mathfrak{g}_p, e)$ might be used to construct integrable hierarchies on Drinfeld-Sokolov reductions of affinizations of truncated current Lie algebras.*

4.3.2 Quantum affine W-algebras as quantum Drinfeld-Sokolov reductions

Let $C^k(\mathfrak{g}_p, e) := V^k(\mathfrak{g}_p) \otimes F(A_p^{ch}) \otimes F(A_p^{ne})$ be the tensor product of the three vertex superalgebras, which is again a vertex superalgebra and contains $V^k(\mathfrak{g}_p)$, $F(A_p^{ch})$ and $F(A_p^{ne})$ as vertex subalgebras. Indeed, $C^k(\mathfrak{g}_p, e)$ can also be considered as the universal enveloping vertex superalgebra of $R^k(\mathfrak{g}_p, e)$.

Let

$$d^p = \sum_{i \in S_p^-} \varepsilon(u_i^*)(u_i + \epsilon(u_i) + (e \mid u_i)_p) - \frac{1}{2} \sum_{i,j \in S_p^-} :\iota([u_i, u_j])\varepsilon(u_i^*)\varepsilon(u_j^*):. \tag{4.26}$$

The element $d^p \in C^k(\mathfrak{g}_p, e)$ is obviously odd. Define a charge grading on $C^k(\mathfrak{g}_p, e)$ by setting

$$\operatorname{cdeg} V^k(\mathfrak{g}_p) = \operatorname{cdeg} F(A_p^{ne}) = 0, \quad -\operatorname{cdeg} \iota(u) = \operatorname{cdeg} \varepsilon(v^*) = 1 \text{ for } u \in \mathfrak{n}_p, v^* \in \mathfrak{n}_p^*.$$

Then it induces a $\mathbb{Z}$-grading on $C^k(\mathfrak{g}_p, e)$ and d^p is of charge degree 1.

Lemma 4.3.5. *We have the following formulas for the λ-bracket of d^p in $C^k(\mathfrak{g}_p, e)$,*

(1) $[d^p{}_\lambda x] = \sum_{i \in S_p^-}([u_i, x] + k(x \mid u_i)_p(\lambda + T))\varepsilon(u_i^*)$ *for $x \in \mathfrak{g}_p$;*

(2) $[d^p{}_\lambda \epsilon(y)] = \sum_{i \in S_p^-}(e \mid [u_i, y])_p \varepsilon(u_i^*)$ *for $y \in \mathfrak{g}_p(-1)$;*

(3) $[d^p{}_\lambda \varepsilon(v^*)] = -\frac{1}{2}\sum_{i,j \in S_p^-}\langle v^*, [u_i, u_j]\rangle \varepsilon(u_i^*)\varepsilon(u_j^*)$ *for $v^* \in \mathfrak{n}_p^*$;*

(4) $[d^p{}_\lambda \iota(u)] = \sum_{i,j \in S_p^-}\langle u_i^*, u\rangle \Big(:\iota([u_i, u_j])\varepsilon(u_j^*): + u_i + (e \mid u_i)_p + \epsilon(u_i)\Big)$ *for $u \in \mathfrak{n}_p$.*

Proof. The proof is the same as that of Lemma 4.3.1, except that we have (4.10) instead of (4.12). But we only need to notice two facts. First, the elements of $R^k(\mathfrak{g}_p, e)$ have same λ-bracket in $C^k(\mathfrak{g}_p, e)$ and in $S(R^k(\mathfrak{g}_p, e))$. Second, the extra term in (4.10) always vanishes in the calculations that we did in the proof of Lemma 4.3.1. Therefore, we have the same result in the end. □

Proposition 4.3.6. *We have $[d^p{}_\lambda d^p] = 0$, which implies that $(d_0^p)^2 = 0$ as d^p is odd.*

Proof. The proof is the same as that of Proposition 4.3.2 for the same reason as in the proof of Lemma 4.3.5. □

Definition 4.3.7. The *quantum affine W-algebra* $W^k(\mathfrak{g}_p, e)$ associated to the data $(\mathfrak{g}_p, e, k)$ is defined to be the cohomology of the complex $(C^k(\mathfrak{g}_p, e), d_0^p)$.

Remark 4.3.8. *When $p = 0$, the complex $(C^k(\mathfrak{g}_p, e), d_0^p)$ is called the BRST complex of the quantum Drinfeld-Sokolov reduction.*

4.3.3 Quantum affine W-algebra as (adjusted) semi-infinite cohomology

Let $\hat{\mathfrak{n}}_p = \mathfrak{n}_p \otimes \mathbb{C}[t, t^{-1}]$ be the affinization of $\mathfrak{n}_p$ with the $\mathbb{Z}$-grading: $(\hat{\mathfrak{n}}_p)_i = \mathfrak{n}_p \otimes t^i$. Denote by $u_{i,n} = u_i \otimes t^n$ for $u_i \in \mathfrak{n}_p$. Then $\{u_{i,n}\}_{n\in\mathbb{Z}, i\in S_p^-}$ forms a basis of $\hat{\mathfrak{n}}_p$. Let $\hat{\mathfrak{n}}_p^* := \mathfrak{n}_p^* \otimes \mathbb{C}[t, t^{-1}]$ and write $u_{i,n}^* = u_i^* \otimes t^n$. Then $\{u_{i,n}^*\}_{n\in\mathbb{Z}, i\in S_p^-}$ forms a basis of $\hat{\mathfrak{n}}_p^*$. One can identify $\hat{\mathfrak{n}}_p^*$ with the restricted dual of $\hat{\mathfrak{n}}_p$ under the pairing $\langle u_{j,m}^*, u_{i,n}\rangle := \delta_{m,-n-1}\delta_{i,j}$, and we define $(\hat{\mathfrak{n}}_p)_i^* = \mathfrak{n}_p^* \otimes t^{-i-1}$.

As $\mathfrak{n}_p$ is nilpotent, by Proposition 3.2.13, $\hat{\mathfrak{n}}_p$ admits a semi-infinite structure through

$$\rho^0(x) = \sum_{i\in S_p^-, n\in\mathbb{Z}} : \iota(\operatorname{ad} x(u_{i,n}))\varepsilon(u_{i,-n-1}^*) : .$$

Let $\beta_e \in \hat{\mathfrak{n}}_p^*$ be defined by $\beta_e(u \otimes t^n) := \delta_{n,-1}(e \mid u)_p$ for $u \in \mathfrak{n}_p$. Then $\beta_e \in (\hat{\mathfrak{n}}_p^*)_{-1}$. Let $\rho^{\beta_e}(x) := \rho^0(x) + \beta_e(x)$ for $x \in \hat{\mathfrak{n}}_p$. Then for $x, y \in \hat{\mathfrak{n}}_p$,

$$\gamma^{\beta_e}(x, y) = [\rho^{\beta_e}(x), \rho^{\beta_e}(y)] - \rho^{\beta_e}([x, y]) = -\beta_e([x, y]).$$

Therefore, $\rho^{\beta_e}(x)$ gives a semi-infinite structure on $\hat{\mathfrak{n}}_p$ if and only if $\beta_e([x, y]) = 0$ for all $x, y \in \mathfrak{n}_p$, which is true if and only if the $\mathbb{Z}$-grading (4.20) is even.

Let $\mathfrak{m}_p = \bigoplus_{i\leq -2} \mathfrak{g}_p(i)$, $\hat{\mathfrak{m}}_p = \mathfrak{m}_p \otimes \mathbb{C}[t, t^{-1}]$ and $\hat{\mathfrak{g}}_p(-1) = \mathfrak{g}_p(-1) \otimes \mathbb{C}[t, t^{-1}]$. Note that $\ker \gamma^{\beta_e} = \hat{\mathfrak{m}}_p$, and $\hat{\mathfrak{g}}_p(-1)$ is a graded complement of $\ker \gamma^{\beta_e}$ in $\hat{\mathfrak{n}}_p$. Moreover, we have $[\hat{\mathfrak{n}}_p, \hat{\mathfrak{n}}_p] \subseteq \hat{\mathfrak{m}}_p$, so the Lie algebra $\hat{\mathfrak{n}}_p$ satisfies the assumption after Remark 3.3.2 with respect to the 1-cochain β_e in the discussion of adjusted semi-infinite cohomology. We can thus consider the adjusted semi-infinite cohomology of $\hat{\mathfrak{n}}_p$ with coefficients in the smooth module $V^k(\mathfrak{g}_p)$, with respect to β_e. The complex is $V^k(\mathfrak{g}_p) \otimes \Lambda^{\infty/2+\bullet}\hat{\mathfrak{n}}_p^* \otimes \mathfrak{F}_{\beta_e}$ and the differential is

$$\begin{aligned} d^{\beta_e} = &\sum_{i\in S_p^-, n\in\mathbb{Z}} u_{i,n}\varepsilon(u_{i,-n-1}^*) - \frac{1}{2} \sum_{\substack{i,j\in S_p^-,\\ n,m\in\mathbb{Z}}} : \iota([u_{i,n}, u_{j,m}])\varepsilon(u_{i,-n-1}^*)\varepsilon(u_{j,-m-1}^*) : \\ &+ \varepsilon(\beta_e) + \sum_{i\in S_p', n\in\mathbb{Z}} \varepsilon(u_{i,-n-1}^*)\epsilon(u_{i,n}). \end{aligned}$$

Recall the Lie (super)algebra structures on $cl(\hat{\mathfrak{n}}_p)$ and $\epsilon(\hat{\mathfrak{g}}_p(-1)) \oplus \mathbb{C}K$ defined in Chapter 3. Note that $\Lambda^{\infty/2+\bullet}\hat{\mathfrak{n}}_p^*$ and $\mathfrak{F}_{\beta_e}$ are the Fock representations of $cl(\hat{\mathfrak{n}}_p)$ and $\epsilon(\hat{\mathfrak{g}}_p(-1)) \oplus \mathbb{C}K$, respectively. Since $cl(\hat{\mathfrak{n}}_p) \cong \widehat{A}_p^{ch}$ and $\epsilon(\hat{\mathfrak{g}}_p(-1)) \oplus \mathbb{C}K \cong \widehat{A}_p^{ne}$, we have $F(A_p^{ch}) \cong \Lambda^{\infty/2+\bullet}\hat{\mathfrak{n}}_p^*$ and $F(A_p^{ne}) \cong \mathfrak{F}_{\beta_e}$ as vector spaces. Therefore, on the complex level, we have $C^k(\mathfrak{g}_p, e) \cong V^k(\mathfrak{g}_p) \otimes \Lambda^{\infty/2+\bullet}\hat{\mathfrak{n}}_p^* \otimes \mathfrak{F}_{\beta_e}$. On the level of differential, $d_0^p = \operatorname{Res}_z d^p(z)$ is the coefficient of z^{-1} in the expression of $d^p(z)$, where $d^p(z)$ is the vertex operator of d^p defined by (4.26) and has the following form,

$$\begin{aligned} d^p(z) = &\sum_{i\in S_p^-} u_i(z)\varepsilon(u_i^*)(z) - \frac{1}{2} \sum_{i,j\in S_p^-} : \iota([u_i, u_j])(z)\varepsilon(u_i^*)(z)\varepsilon(u_j^*)(z) : \\ &+ \sum_{i\in S_p^-} (e \mid u_i)\varepsilon(u_i^*)(z) + \sum_{i\in S_p'} \varepsilon(u_i^*)(z)\epsilon(u_i)(z). \end{aligned}$$

Recall the expressions of the vertex operators $u_i(z), \varepsilon(u_i^*)(z), \iota(u_i)(z)$ and $\epsilon(u_i)(z)$ given in the examples (4.2.12) and (4.2.14). We have

$$\mathrm{Res}_z \sum_{i\in S_p^-} u_i(z)\varepsilon(u_i^*)(z) = \mathrm{Res}_z \sum_{\substack{i\in S_p^-,\\ m,n\in\mathbb{Z}}} u_{i,n}\varepsilon(u_{i,m}^*)z^{-n-m-2} = \sum_{i\in S_p^-,n\in\mathbb{Z}} u_{i,n}\varepsilon(u_{i,-n-1}^*),$$

$$\mathrm{Res}_z \sum_{i\in S_p^-} (e \mid u_i)\varepsilon(u_i^*)(z) = \mathrm{Res}_z \sum_{i\in S_p^-,n\in Z} (e\mid u_i)_p \varepsilon(u_{i,n}^*)z^{-n-1} = \sum_{i\in S_p^-} (e\mid u_i)_p\varepsilon(u_{i,0}^*) = \varepsilon(\beta_e),$$

and

$$\mathrm{Res}_z \sum_{i\in S_p'} \varepsilon(u_i^*)(z)\epsilon(u_i)(z) = \mathrm{Res}_z \sum_{\substack{i\in S_p^-,\\ m,n\in\mathbb{Z}}} \varepsilon(u_{i,m}^*)\epsilon(u_{i,n})z^{-n-m-2} = \sum_{i\in S_p',n\in\mathbb{Z}} \varepsilon(u_{i,-n-1}^*)\epsilon(u_{i,n}).$$

Let $X = \sum_{i,j\in S_p^-} : \iota([u_i,u_j])(z)\varepsilon(u_i^*)(z)\varepsilon(u_j^*)(z) :$. Then

$$\begin{aligned}\mathrm{Res}_z X &= \mathrm{Res}_z \sum_{\substack{i,j\in S_p^-,\\ n,m,\ell\in\mathbb{Z}}} : \iota([u_i,u_j]\otimes t^\ell)\varepsilon(u_{i,-m-1}^*)\varepsilon(u_{j,-n-1}^*) : z^{m+n-\ell-1} \\ &= \sum_{\substack{i,j\in S_p^-,\\ n,m\in\mathbb{Z}}} : \iota([u_{i,n},u_{j,m}])\varepsilon(u_{i,-n-1}^*)\varepsilon(u_{j,-m-1}^*) : .\end{aligned}$$

Therefore, we have $d_0^p = d^{\beta_e}$. This proves the following theorem.

Theorem 4.3.9 ([He17a]). *The affine W-algebra $W^k(\mathfrak{g}_p, e)$ is the adjusted semi-infinite cohomology of $\hat{\mathfrak{n}}_p$ with coefficients in $V^k(\mathfrak{g}_p)$ with respect to β_e. By Proposition 3.3.12, it is also an ordinary semi-infinite cohomology, i.e.,*

$$W^k(\mathfrak{g}_p, e) = H_a^{\infty/2+\bullet}(\hat{\mathfrak{n}}_p, \beta_e, V^k(\mathfrak{g}_p)) \cong H^{\infty/2+\bullet}(\hat{\mathfrak{n}}_p, V^k(\mathfrak{g}_p)\otimes \mathfrak{F}_{\beta_e})$$

Remark 4.3.10. *When the $\mathbb{Z}$-grading in (4.20) is even and $p=0$, i.e., when ρ^{β_e} gives a semi-infinite structure on $\hat{\mathfrak{n}}_p$, the Fock representation $\mathfrak{F}_{\beta_e}$ reduces to a one-dimensional module $\mathbb{C}_{\beta_e}$ on which $x\in\hat{\mathfrak{n}}_p$ acts as $\beta_e(x)$. This recovers the semi-infinite cohomology realization of affine W-algebras in principal nilpotent cases [FF90].*

Remark 4.3.11. *When $p=0$, the isomorphism $W^k(\mathfrak{g}_p,e)\cong H^{\infty/2+\bullet}(\hat{\mathfrak{n}}_p, V^k(\mathfrak{g}_p)\otimes\mathfrak{F}_{\beta_e})$ was also observed in [Ara05] (Remark 3.6.1), though the construction there was a bit different from ours.*

Chapter 5

Higher level Zhu algebras

We prove that higher level Zhu algebras of a vertex operator algebra are isomorphic to subquotients of its universal enveloping algebra. The main results of this chapter are contained in [He17b].

5.1 The Zhu algebra and higher level Zhu algebras

In this section, we briefly recall the definitions, mainly for fixing the notation. For details, we refer to the papers [DLM98, FHL93, Zhu96].

5.1.1 Vertex operator algebras and their modules

Definition 5.1.1. A *vertex operator algebra* is a vertex algebra $(V, |0\rangle, Y, T)$ with a *conformal vector* or a *Virasoro element* ω, such that if we write $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$, i.e., $L(n) = \omega_{n+1}$, then

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{c}{12}(m^3 - m)\delta_{m,-n}$$

for some $c \in \mathbb{C}$, which is called the *central charge* of V. Moreover, $L(-1) = T$ is the infinitesimal translation operator and $L(0)$ is diagonalizable on V, which gives V a $\mathbb{Z}$-grading $V = \bigoplus_{n\in\mathbb{Z}} V_n$ with $L(0)|_{V_n} = n\mathrm{Id}\, V_n$, $\dim\, V_n < \infty$ for all $n \in \mathbb{Z}$ and $V_n = 0$ for $n \ll 0$.

An element $v \in V_n$ is called homogeneous of *conformal weight* n, and we denote it by Δ_v. Whenever we use the notation Δ_v, we assume that v is homogeneous.

Definition 5.1.2. A *weak module* for a vertex operator algebra V is a vector space M, with a linear map $Y_M : V \to \mathrm{End}\, M[[z, z^{-1}]]$ sending v to $Y_M(v, z) = \sum v_n^M z^{-n-1}$ and satisfying:

(1) $Y_M(|0\rangle, z) = \mathrm{Id}_M$ and $Y_M(v, z)w \in M((z))$ for all $v \in V, w \in M, $, i.e., $v_n^M w = 0$ for $n \ll 0$.

(2) For all $\ell, m, n \in \mathbb{Z}$ and $u, v \in V$, we have the Jacobi identity

$$\sum_{i\geq 0}(-1)^i\binom{\ell}{i}\left(u^M_{m+\ell-i}v^M_{n+i} - (-1)^\ell v^M_{n+\ell-i}u^M_{m+i}\right) = \sum_{i\geq 0}\binom{m}{i}(u_{\ell+i}v)^M_{m+n-i}.$$

A weak module M is called *admissible* if it has a $\mathbb{Z}_{\geq 0}$-grading $M = \bigoplus_{n\geq 0} M_n$ and satisfies:

(3) For any homogeneous element $v \in V$, we have

$$v_n^M M_m \subseteq M_{m+\Delta_v-n-1}.$$

Submodules, quotient modules, simple modules and semi-simple modules can be defined in the obvious way.

5.1.2 The Zhu algebra

Let $(V, Y, |0\rangle, \omega)$ be a vertex operator algebra. Following [Zhu96], we will construct an associative algebra $\mathrm{Zhu}(V)$ associated to V.

Let

$$O(V) := \mathrm{span}\{u \circ v \mid u, v \in V\},$$

where the linear product $\circ$ is defined on homogeneous $u \in V$ by

$$u \circ v := \mathrm{Res}_z \left(Y(u, z)v \frac{(1+z)^{\Delta_u}}{z^2} \right) = \sum_{i\geq 0} \binom{\Delta_u}{i} u_{i-2}v.$$

Define a product $*$ on V by the formula:

$$u * v := \mathrm{Res}_z \left(Y(u, z)v \frac{(1+z)^{\Delta_u}}{z} \right) = \sum_{i\geq 0} \binom{\Delta_u}{i} u_{i-1}v.$$

The subspace $O(V)$ is known to be a two-sided ideal of V under $*$ [Zhu96].

Let

$$\mathrm{Zhu}(V) := V/O(V).$$

Theorem 5.1.3. *[Zhu96]. The product $*$ induces an associative algebra structure on $\mathrm{Zhu}(V)$ with identity $|0\rangle + O(V)$.*

For an admissible V-module $M = \bigoplus_{n\geq 0} M_n$, we call M_n the n-th level and M_0 the top level of M. Denote by $o^M(u) := u^M_{\Delta_u-1}$ for all homogeneous $u \in V$ and extend linearly to V. Then $o^M(u)M_n \subseteq M_n$. In particular, $o^M(u)$ preserves the top level. Moreover, the identities

$$o^M(u)o^M(v) = o^M(u * v) \;\text{ and }\; o^M(u') = 0$$

hold for all $u, v \in V$ and $u' \in O(V)$ when restricted to the top level M_0. Thus, the top level M_0 is a $\mathrm{Zhu}(V)$-module under the action $(u + O(V)) \cdot m = o^M(u)m$.

The correspondence $M \mapsto M_0$ gives a functor, which we denote by Ω_0, from the category of admissible V-modules to the category of $\mathrm{Zhu}(V)$-modules. On the other hand, Zhu constructed another functor L^0 from the category of $\mathrm{Zhu}(V)$-modules to the category of admissible V-modules in his thesis paper [Zhu96]. Given a $\mathrm{Zhu}(V)$-module U with action π, $L^0(U)$ is an admissible module for V with top level being U. Moreover, we have $\pi(v)m = o^{L^0(U)}(v)m$ for all $m \in U$ and $v \in V$.

Theorem 5.1.4. *[Zhu96]. The two functors Ω_0, L^0 are mutually inverse to each other when restricted to the full subcategory of completely reducible admissible V-modules and the full subcategory of completely reducible $\mathrm{Zhu}(V)$-modules.*

5.1.3 Higher level Zhu algebras

Let $(V, Y, |0\rangle, \omega)$ be a vertex operator algebra. Following [DLM98], we are going to construct an associative algebra $A_n(V)$ for each nonnegative integer n, which we will call the *level n Zhu algebra*[1], with $A_0(V)$ being exactly the Zhu algebra $\mathrm{Zhu}(V)$. We will call the algebras $A_n(V)$ *higher level Zhu algebras* when $n \geq 1$.

Recall that $L(n) = \omega_{n+1}$, where ω is the Virasoro element of V. For $n \geq 0$, let

$$O_n(V) := \mathrm{span}\{u \circ_n v, L(-1)u + L(0)u \mid u, v \in V\},$$

where the linear product $\circ_n$ is defined on homogeneous $u \in V$ by

$$\begin{aligned} u \circ_n v : &= \mathrm{Res}_z \left(Y(u,z)v \frac{(1+z)^{\Delta_u + n}}{z^{2n+2}} \right) \\ &= \sum_{i=0}^{\infty} \binom{\Delta_u + n}{i} u_{i-2n-2} v. \end{aligned}$$

Define a product $*_n$ on V by the formula:

$$\begin{aligned} u *_n v &:= \sum_{m=0}^{n} (-1)^m \binom{m+n}{n} \mathrm{Res}_z \left(Y(u,z)v \frac{(1+z)^{\Delta_u + n}}{z^{n+m+1}} \right) \\ &= \sum_{m=0}^{n} \sum_{i=0}^{\infty} (-1)^m \binom{m+n}{n} \binom{\Delta_u + n}{i} u_{i-m-n-1} v. \end{aligned}$$

The subspace $O_n(V)$ is a two-sided ideal of V under $*_n$ [DLM98].

Let

$$A_n(V) := V/O_n(V).$$

Theorem 5.1.5. *[DLM98]. The product $*_n$ induces an associative algebra structure on $A_n(V)$ with identity $|0\rangle + O_n(V)$. Moreover, the identity map on V induces a surjective algebra homomorphism from $A_n(V)$ to $A_{n-1}(V)$ for $n \geq 1$.*

[1] We follow the terminology as in [vE11] for the twisted case.

Remark 5.1.6. *Note that $L(-1)u + L(0)u = u \circ |0\rangle$ and $u \circ_0 v = u \circ v$, so $O_0(V)$ coincides with $O(V)$. Moreover, as $u *_0 v = u * v$, the algebra $A_0(V) = \mathrm{Zhu}(V)$ is just the Zhu algebra.*

We have an inverse system of associative algebras:

$$A_0(V) \twoheadleftarrow A_1(V) \twoheadleftarrow \cdots \twoheadleftarrow A_n(V) \twoheadleftarrow A_{n+1}(V) \twoheadleftarrow \cdots . \tag{5.1}$$

These higher level Zhu algebras play similar roles to that of the Zhu algebra in the representation theory of vertex operator algebras. To describe the relationship between the representations of $A_n(V)$ and those of V, we recall a Lie algebra associated to V.

Consider the vector space $V \otimes \mathbb{C}[t, t^{-1}]$ and the linear operator

$$\partial := L(-1) \otimes \mathrm{Id} + \mathrm{Id} \otimes \frac{d}{dt}.$$

Let

$$\hat{V} := \frac{V \otimes \mathbb{C}[t, t^{-1}]}{\partial(V \otimes \mathbb{C}[t, t^{-1}])}.$$

Denote by $v(m)$ the image of $v \otimes t^m$ in $\hat{V}$ for $v \in V$ and $m \in \mathbb{Z}$. The vector space $\hat{V}$ is a $\mathbb{Z}$-graded Lie algebra by defining the degree of $v(m)$ to be $\Delta_v - m - 1$ and the Lie bracket:

$$[u(m), v(n)] = \sum_{i \geq 0} \binom{m}{i} (u_i v)(m + n - i) \text{ for } u, v \in V. \tag{5.2}$$

As the Lie bracket (5.2) in $\hat{V}$ is just the commutator formula (4.2) in V, the natural map from $\hat{V}$ to $\mathrm{End}\, V$ sending $v(m)$ to v_m is a Lie algebra homomorphism. In this way, we can consider a V-module as a $\hat{V}$-module.

Denote the homogeneous subspace of $\hat{V}$ of degree m by $\hat{V}(m)$. Then $\hat{V}(0)$ is a Lie subalgebra of $\hat{V}$. Consider the Lie algebra structure of $A_n(V)$ with Lie bracket $[u, v] = u *_n v - v *_n u$ for $u, v \in V$. One can show that [DLM98] there is a surjective Lie algebra homomorphism from $\hat{V}(0)$ to $A_n(V)$ for each n, sending $o(v) := v(\Delta_v - 1)$ to $v + O_n(V)$. Let $U(\hat{V})$ be the universal enveloping algebra of $\hat{V}$. Then it inherits a natural $\mathbb{Z}$-grading from $\hat{V}$, say $U(\hat{V}) = \bigoplus_{n \in \mathbb{Z}} U(\hat{V})_n$.

Let $P_n = \bigoplus_{i > n} \hat{V}(i) \oplus \hat{V}(0)$. Given an $A_n(V)$-module N, we can consider it as a $\hat{V}(0)$-module, and then as a P_n-module by letting $\bigoplus_{i > n} \hat{V}(i)$ act trivially. Define

$$M_n(N) = \mathrm{Ind}_{P_n}^{\hat{V}}(N) = U(\hat{V}) \otimes_{U(P_n)} N.$$

By setting the degree of N to be n, the $\mathbb{Z}$-gradation of $\hat{V}$ lifts to $M_n(N)$ with $M_n(N)(i) = U(\hat{V})_{i-n} N$. Let W be the subspace of $M_n(N)$ spanned by the coefficients of (where $u, v \in V, m \in M_n(N)$)

$$(z + w)^{\Delta_u + n} Y(u, z + w) Y(v, w) m - (w + z)^{\Delta_u + n} Y(Y(u, z)v, w) m.$$

Let

$$\overline{M}_n(N) := M_n(N) / U(\hat{V}) W. \tag{5.3}$$

Theorem 5.1.7 ([DLM98]). *The space $\overline{M}_n(N) = \sum_{i\geq 0} \overline{M}_n(N)(i)$ admits an admissible V-module structure with $\overline{M}_n(N)(0) \neq 0$ and $\overline{M}_n(N)(n) = N$.*

Let $M = \bigoplus_{i\geq 0} M_i$ be an admissible V-module and n a nonnegative integer, and define the subspace

$$\Omega_n(M) := \{m \in M \mid \hat{V}(-k)m = 0 \text{ if } k > n\}.$$

Then one can show that $\Omega_n(M)$ admits an $A_n(V)$-module structure under the action $v \cdot m = o^M(v)m$, with each M_i being a submodule for $0 \leq i \leq n$. The module $\overline{M}_n(N)$ has the universal property that if W is any weak V-module, and $\varphi : N \to \Omega_n(W)$ any $A_n(V)$-module homomorphism, then there is a unique V-module homomorphism $\tilde{\varphi} : \overline{M}_n(N) \to W$ which extends φ [DLM98].

Since there is a surjective homomorphism $A_n(V) \twoheadrightarrow A_{n-1}(V)$, the subspace $\Omega_{n-1}(M) \subseteq \Omega_n(M)$ is naturally an $A_n(V)$-module. Let

$$\Omega_n/\Omega_{n-1}(M) := \frac{\Omega_n(M)}{\Omega_{n-1}(M)}.$$

Then Ω_n/Ω_{n-1} defines a functor from the category of admissible V-modules to the category of $A_n(V)$-modules. The good thing is that this functor has an inverse when restricted to an appropriate subcategory. In [DLM98], the authors constructed a functor L^n from the category of $A_n(V)$-modules to the category of admissible V-modules, such that, for a given $A_n(V)$-module N with action π, if N itself and its proper submodules do not factor through $A_{n-1}(V)$ (this condition was added in [BVY17]), then $\Omega_n/\Omega_{n-1}(L^n(N)) \cong N$ as $A_n(V)$-modules, i.e., $o^{L^n(U)}(v)m = \pi(v)m$ for all $v \in V$ and $m \in N$.

Theorem 5.1.8. *[DLM98, BVY17]. The functors Ω_n/Ω_{n-1} and L^n are inverse to each other when restricted to the full subcategory of completely reducible admissible V-modules that are generated by their degree n subspace and the full subcategory of completely reducible $A_n(V)$-modules whose irreducible components do not factor through $A_{n-1}(V)$.*

5.2 The universal enveloping algebra and its subquotients

To define the universal enveloping algebra of a vertex operator algebra, we need to introduce a completion notation, as the Jacobi identity contains infinite sums.

Recall that the Lie algebra $\hat{V}$ that we constructed in the previous section is $\mathbb{Z}$-graded. The zero component $U(\hat{V})_0$ of $U(\hat{V})$ contains $U(\hat{V}(0))$, the universal enveloping algebra of $\hat{V}(0)$, as a subalgebra. For $n \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\leq 0}$, let

$$U(\hat{V})_n^k = \sum_{i\leq k} U(\hat{V})_{n-i} U(\hat{V})_i \quad \text{and} \quad U(\hat{V}(0))^k = U(\hat{V}(0)) \cap U(\hat{V})_0^k.$$

Then

$$\cdots \subseteq U(\hat{V})_n^k \subseteq U(\hat{V})_n^{k+1} \subseteq \cdots \subseteq U(\hat{V})_n^0 = U(\hat{V})_n$$

and

$$\cdots \subseteq U(\hat{V}(0))^k \subseteq U(\hat{V}(0))^{k+1} \subseteq \cdots \subseteq U(\hat{V}(0))^0 = U(\hat{V}(0))$$

are well-defined filtrations of $U(\hat{V})_n$ and $U(\hat{V}(0))$, respectively. Moreover, we have

$$\bigcap_k U(\hat{V})_n^k = 0, \qquad \bigcup_k U(\hat{V})_n^k = U(\hat{V})_n.$$

Hence, the filtration $\{U(\hat{V})_n^k\}_{k\leq 0}$ forms a fundamental neighborhood system of $U(\hat{V})_n$. Let $\tilde{U}(\hat{V})_n$ be the completion of $U(\hat{V})_n$ with respect to this filtration, i.e., infinite sums are allowed in $\tilde{U}(\hat{V})_n$, and for any given k, only finitely many terms are contained in $U(\hat{V})_n^{k+1} \setminus U(\hat{V})_n^k$. Let $\tilde{U}(\hat{V}(0))$ be the completion of $U(\hat{V}(0))$ with respect to the filtration $\{U(\hat{V}(0))^k\}_{k\leq 0}$. It is obviously a subspace of $\tilde{U}(\hat{V})_0$.

Let

$$\tilde{U}(\hat{V}) := \bigoplus_{n\in\mathbb{Z}} \tilde{U}(\hat{V})_n.$$

The space $\tilde{U}(\hat{V})$ becomes a $\mathbb{Z}$-graded ring with each component $\tilde{U}(\hat{V})_n$ being complete. The subspace $U(\hat{V})$ is a dense subalgebra of $\tilde{U}(\hat{V})$ with $U(\hat{V})_n$ being dense in $\tilde{U}(\hat{V})_n$ for all n. The completion $\tilde{U}(\hat{V})$ is called a degreewise completed topological ring in the theory of quasi-finite algebras studied by A. Matsuo et al. in [MNT10].

Consider the relations

$$\begin{aligned}
\langle \text{Vac} \rangle : &\quad |0\rangle(i) = \delta_{i,-1}, \text{ for all } i \in \mathbb{Z}, \\
\langle \text{Vir} \rangle : &\quad [L(m), L(n)] = (m-n)L(m+n) + \delta_{m+n,0}\frac{m^3-m}{12}\mathbf{c}, \text{ for all } m,n \in \mathbb{Z}, \\
J^{u,v}_{m,n,\ell} : &\quad \sum_{i\geq 0}(-1)^i \binom{\ell}{i} \Big(u(m+\ell-i)v(n+i) - (-1)^\ell v(n+\ell-i)u(m+i)\Big) \\
&\quad = \sum_{i\geq 0}\binom{m}{i}(u_{\ell+i}v)(m+n-i), \text{ for } u,v \in V \text{ and } m,n,\ell \in \mathbb{Z}.
\end{aligned}$$

Remark 5.2.1. *The element $L(n)$ should be considered as the image of $\omega \otimes t^{n+1}$ in $\hat{V}$. The Jacobi relation $J^{u,v}_{m,n,\ell}$ is now well-defined in $\tilde{U}(\hat{V})$.*

Definition 5.2.2. The *universal enveloping algebra* $U(V)$ of V is the quotient of $\tilde{U}(\hat{V})$ by the relations: $\langle\text{Vac}\rangle$, $\langle\text{Vir}\rangle$ and $\langle J^{u,v}_{m,n,\ell} \mid u,v \in V, m,n,\ell \in \mathbb{Z}\rangle$.

Remark 5.2.3. *The universal enveloping algebra $U(V)$ of a vertex operator algebra V is an associative algebra, while the universal enveloping vertex algebra $V(R)$ of a non-linear Lie conformal algebra R that we defined in Definition 4.2.8 is a vertex algebra.*

All the relations $\langle\text{Vac}\rangle$, $\langle\text{Vir}\rangle$ and $J^{u,v}_{m,n,\ell}$ are homogeneous, so the universal enveloping algebra $U(V)$ inherits a natural $\mathbb{Z}$-grading from $\tilde{U}(\hat{V})$.

The image of $\tilde{U}(\hat{V}(0))$ in $U(V)$ is obviously contained in $U(V)_0$, which we denote by $U(V(0))$ and is a subalgebra of $U(V)$.

Let

$$U(V)_0^k := \sum_{i \leq k} U(V)_{-i} U(V)_i \quad \text{and} \quad U(V(0))^k := U(V(0)) \cap U(V)_0^k.$$

Then $\dfrac{U(V)_0}{U(V)_0^k}$ and $\dfrac{U(V(0))}{U(V(0))^k}$ inherit associative algebra structures, as $U(V)_0^k$ and $U(V(0))^k$ are two-sided ideals of $U(V)_0$ and $U(V(0))$, respectively. By the obvious inclusions $U(V)_0^k \subseteq U(V)_0^{k+1}$ and $U(V(0))^k \subseteq U(V(0))^{k+1}$, we have two inverse systems of algebras:

$$\frac{U(V)_0}{U(V)_0^{-1}} \twoheadleftarrow \frac{U(V)_0}{U(V)_0^{-2}} \twoheadleftarrow \cdots \twoheadleftarrow \frac{U(V)_0}{U(V)_0^{-n}} \twoheadleftarrow \frac{U(V)_0}{U(V)_0^{-n-1}} \twoheadleftarrow \cdots,$$
$$\frac{U(V(0))}{U(V(0))^{-1}} \twoheadleftarrow \frac{U(V(0))}{U(V(0))^{-2}} \twoheadleftarrow \cdots \twoheadleftarrow \frac{U(V(0))}{U(V(0))^{-n}} \twoheadleftarrow \frac{U(V(0))}{U(V(0))^{-n-1}} \twoheadleftarrow \cdots.$$

Our goal is prove that these two inverse systems of associative algebras are both isomorphic to the inverse system given by higher level Zhu algebras (5.1). More precisely, we are going to prove that

$$A_n(V) \cong \frac{U(V)_0}{U(V)_0^{-n-1}} \cong \frac{U(V(0))}{U(V(0))^{-n-1}} \text{ for } n \geq 0.$$

5.3 The isomorphisms

One of our motivations for this study is the paper [FZ92] of I. Frenkel and Y. C. Zhu, where they observed that the Zhu algebra is isomorphic to a subquotient of the universal enveloping algebra. In this section, we prove that all higher level Zhu algebras are also isomorphic to subquotients of the universal enveloping algebra.

For simplicity, we use the following notation: For $u, v \in V$ and $m, n, \ell \in \mathbb{Z}$, let

$$\begin{aligned} {}^1J^{u,v}_{m,n,\ell} &:= \sum_{i \geq 0} \binom{m}{i} (u_{\ell+i} v)(m+n-i), \\ {}^2J^{u,v}_{m,n,\ell} &:= \sum_{i \geq 0} (-1)^i \binom{\ell}{i} \left(u(m+\ell-i)v(n+i) - (-1)^\ell v(n+\ell-i)v(m+i) \right). \end{aligned}$$

They are just the two sides of the Jacobi identity $J^{u,v}_{m,n,\ell}$, so in the universal enveloping algebra $U(V)$, we have ${}^1J^{u,v}_{m,n,\ell} = {}^2J^{u,v}_{m,n,\ell}$.

We use the following notation, which is defined for homogeneous elements and extended linearly to all of V.

$$J_n(u) := u(\Delta_u - 1 + n).$$

A good property of this notation is that the degree of $J_n(u)$ is always $-n$.

Let

$$
\begin{aligned}
{}^{(1)}J^{u,v}_{m,n,\ell} &:= {}^{1}J^{u,v}_{m+\Delta_u-1,n+\Delta_v-1,\ell} \\
&= \sum_{i\geq 0}\binom{m+\Delta_u-1}{i} J_{m+n+\ell}(u_{\ell+i}v), \\
{}^{(2)}J^{u,v}_{m,n,\ell} &:= {}^{2}J^{u,v}_{m+\Delta_u-1,n+\Delta_v-1,\ell} \\
&= \sum_{i\geq 0}(-1)^i\binom{\ell}{i}(J_{m+\ell-i}(u)J_{n+i}(v)-(-1)^{\ell}J_{n+\ell-i}(v)J_{m+i}(u)).
\end{aligned}
$$

Every term in the expressions ${}^{(1)}J^{u,v}_{m,n,\ell}$, ${}^{(2)}J^{u,v}_{m,n,\ell}$ is of the same degree $-m-n-\ell$.

For a negative integer n and a positive integer k, recall that

$$
\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = (-1)^k\binom{-n+k-1}{k}. \tag{5.4}
$$

The statement of the following lemma was suggested by Atsushi Matsuo.

Lemma 5.3.1. *For any integers s, t and N satisfying $N+s\geq 0$,*

$$
\begin{aligned}
X :=& \sum_{j=0}^{N}\binom{-N-s-1}{j}\,{}^{(2)}J^{u,v}_{N+1,t+j,-N-s-1-j} \\
=& J_{-s}(u)J_t(v) + \sum_{k\geq N+1}\sum_{j=0}^{N}(-1)^j\binom{N+s+j}{j}\binom{N+s-k}{k-j}J_{-k-s}(u)J_{k+t}(v) \\
&- \sum_{j=0}^{N}\sum_{i\geq 0}(-1)^{N+s+1}\binom{N+s+j}{j}\binom{N+s+j+i}{i}J_{t-N-s-1-i}(v)J_{N+1+i}(u).
\end{aligned}
$$

Proof. By definition, ${}^{(2)}J^{u,v}_{N+1,t+j,-N-s-1-j} = A - B$, where

$$
\begin{aligned}
A &= \sum_{i\geq 0}(-1)^i\binom{-N-s-1-j}{i}J_{-s-j-i}(u)J_{t+j+i}(v), \\
B &= \sum_{i\geq 0}(-1)^{-N-s-1-j+i}\binom{-N-s-1-j}{i}J_{t-N-s-1-i}(v)J_{N+1+i}(u).
\end{aligned}
$$

Therefore, $X = C - D$, where

$$\begin{aligned}
C &= \sum_{j=0}^{N} \binom{-N-s-1}{j} A \\
&= \sum_{j=0}^{N} \sum_{i\geq 0} (-1)^j \binom{N+s+j}{j} \binom{N+s+j+i}{i} J_{-s-j-i}(u) J_{t+j+i}(v), \\
D &= \sum_{j=0}^{N} \binom{-N-s-1}{j} B \\
&= \sum_{j=0}^{N} \sum_{i\geq 0} (-1)^{N+s+1} \binom{N+s+j}{j} \binom{N+s+i+j}{i} J_{t-N-s-1-i}(v) J_{N+1+i}(u).
\end{aligned}$$

We used the formula (5.4) in the above calculation.

Let $k = i + j$ in the expression of C. Then

$$\begin{aligned}
C &= \sum_{j=0}^{N} \sum_{k\geq j} (-1)^j \binom{N+s+j}{j} \binom{N+s+k}{k-j} J_{-s-k}(u) J_{k+t}(v) \\
&= \sum_{k=0}^{N} \sum_{j=0}^{k} (-1)^j \binom{N+s+j}{j} \binom{N+s+k}{k-j} J_{-s-k}(u) J_{k+t}(v) \\
&\quad + \sum_{k\geq N+1} \sum_{j=0}^{N} (-1)^j \binom{N+s+j}{j} \binom{N+s+k}{k-j} J_{-s-k}(u) J_{k+t}(v).
\end{aligned} \tag{5.5}$$

In the expression (5.5), for $1 \leq k \leq N$, we have

$$\begin{aligned}
&\sum_{j=0}^{k} (-1)^j \binom{N+s+j}{j} \binom{N+s+k}{k-j} J_{-s-k}(u) J_{k+t}(v) \\
&\quad = \sum_{j=0}^{k} (-1)^j \frac{(N+s+j)!}{j!(N+s)!} \frac{(N+s+k)!}{(k-j)!(N+s+j)!} J_{-s-k}(u) J_{k+t}(v) \\
&\quad = \sum_{j=0}^{k} (-1)^j \frac{(N+s+k)!}{(N+s)!k!} \frac{k!}{(k-j)!j!} J_{-s-k}(u) J_{k+t}(v) \\
&\quad = \binom{N+s+k}{k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} J_{-s-k}(u) J_{k+t}(v) \\
&\quad = 0,
\end{aligned}$$

and for $k = 0$, we will have $j = 0$, so only one term will be left in (5.5), namely, $J_{-s}(u)J_t(b)$. □

Corollary 5.3.2. In the universal enveloping algebra $U(V)$, for any integers s, t and N satisfying

$N+s\geq 0$, we have the identity

$$
\begin{aligned}
&J_{-s}(u)J_t(v)\\
&=\sum_{j=0}^{N}\sum_{i\geq 0}(-1)^i\binom{N+\Delta_u}{i}\binom{-N-s-1}{j}J_{t-s}(u_{-N-s-i-j-1}v)\\
&\quad-\sum_{k\geq N+1}\sum_{j=0}^{N}(-1)^j\binom{N+s+j}{j}\binom{N+s-k}{k-j}J_{-k-s}(u)J_{k+t}(v)\\
&\quad+\sum_{j=0}^{N}\sum_{i\geq 0}(-1)^{N+s+1}\binom{N+s+j}{j}\binom{N+s+j+i}{i}J_{t-N-s-1-i}(v)J_{N+1+i}(u).
\end{aligned}
$$

Proof. In the universal enveloping algebra, we have

$$
\begin{aligned}
&\sum_{j=0}^{N}\binom{-N-s-1}{j}{}^{(2)}J^{u,v}_{N+1,t+j,-N-s-1-j}\\
&=\sum_{j=0}^{N}\binom{-N-s-1}{j}{}^{2}J^{u,v}_{N+1+\Delta_u,t+j+\Delta_v,-N-s-1-j}\\
&=\sum_{j=0}^{N}\binom{-N-s-1}{j}{}^{1}J^{u,v}_{N+1+\Delta_u,t+j+\Delta_v,-N-s-1-j}\\
&=\sum_{j=0}^{N}\sum_{i\geq 0}(-1)^i\binom{-N-s-1}{j}\binom{N+\Delta_u}{i}J_{t-s}(u_{-N-s-i-j-1}v).
\end{aligned}
$$

The desired identity then follows from Lemma 5.3.1. □

The following lemma will be very important in the proof of Theorem 5.3.4.

Lemma 5.3.3. *Let $n\geq 0$. Then every element $\sum J_{n_1}(u_1)\cdots J_{n_m}(u_m)$ in $\dfrac{U(V)_0}{U(V)_0^{-n}}$ can be expressed as $J_0(u(w))$ for some $u(w)\in V$.*

Proof. We only need to prove the claim for monomials $w=J_{n_1}(u_1)\cdots J_{n_m}(u_m)$. Define the degree of w to be m, i.e., the number of factors of it. Then a degree one element in $U(V)_0$ is just an element of the form $J_0(u)$ for some $u\in V$, and we need to show that every monomial in the quotient $\dfrac{U(V)_0}{U(V)_0^{-n}}$ is congruent to a degree one element.

We use induction on the degree of the monomial w. If $m=1$, there is nothing to do. Let $m=k\geq 2$ and assume that for every monomial of degree less than k, it is congruent to a degree one element in the quotient $\dfrac{U(V)_0}{U(V)_0^{-n}}$.

Use the formula in Corollary 5.3.2 for $J_{n_{m-1}}(u_{m-1})J_{n_m}(u_m)$, where

$$-s=n_{m-1},\ t=n_m,\ u=u_{m-1},\ v=u_m.$$

In the statement of Corollary 5.3.2, choose N sufficiently large, so that $\min\{N+n_m, N\} > n$. Then $J_{k+n_m}(u_m)$ and $J_{N+1+i}(u_{m-1})$ are both contained in $\bigoplus_{j\le -n} U(V)_j$ for $k \ge N+1$, and $w = J_{n_1}(u_1)\cdots J_{n_m}(u_m)$ is congruent to a linear combination of the following lower degree monomials:

$$J_{n_1}(u_1)\cdots J_{n_{m-2}}(u_{m-2})J_{n_m+n_{m-1}}((u_{m-1})_{-N+n_{m-1}-i-j-1}u_m).$$

By induction, these lower degree monomials are congruent to degree one monomials, so w is itself congruent to a degree one monomial. □

Now we are in a position to prove the isomorphisms between higher level Zhu algebras and subquotients of the universal enveloping algebra.

Theorem 5.3.4. *For $n \ge 0$, we have the isomorphism*

$$A_n(V) \cong \frac{U(V)_0}{U(V)_0^{-n-1}}. \tag{5.6}$$

Proof. Let φ be the map from V to $U(V)_0$ sending v to $o(v)$, where $o(v)$ is the image of $v(\Delta_v - 1)$ in $U(V)$ for homogeneous v and extended linearly to V. Combine it with the canonical quotient map from $U(V)_0$ to $\dfrac{U(V)_0}{U(V)_0^{-n-1}}$. Then Lemma 5.3.3 tells us that this map is surjective.

First, we show that φ factors through $A_n(V)$, i.e., $\varphi(O_n(V)) \subseteq U(V)_0^{-n-1}$.

Recall that $O_n(V) = \operatorname{span}\{u \circ_n v, L(-1)u + L(0)u \mid u, v \in V, u \text{ homogeneous}\}$, where

$$u \circ_n v = \sum_{i=0}^{\Delta_u+n} \binom{\Delta_u+n}{i} u_{i-2n-2}v.$$

As $\varphi(L(-1)u + L(0)u) \equiv 0$, we only need to prove that $\varphi(u \circ_n v) \in U(V)_0^{-n-1}$. Assume that u, v are both homogeneous. Then $\Delta_{u_{i-2n-2}v} = \Delta_u + \Delta_v + 2n + 1 - i$, and

$$\begin{aligned}
\varphi(u \circ_n v) &= \sum_{i=0}^{\Delta_u+n} \binom{\Delta_u+n}{i} (u_{i-2n-2}v)(\Delta_u + \Delta_v + 2n - i) \\
&= {}^{(1)} J^{u,v}_{n+1,n+1,-2n-2} \\
&= {}^{(2)} J^{u,v}_{n+1,n+1,-2n-2} \\
&= \sum_{i\ge 0} (-1)^i \binom{-2n-2}{i} J_{-n-1-i}(u) J_{n+1+i}(v) \\
&\qquad - \sum_{i\ge 0} (-1)^i \binom{-2n-2}{i} J_{-n-1-i}(v) J_{n+1+i}(u).
\end{aligned}$$

As $\deg\ J_{n+1+i}(v) = \deg\ J_{n+1+i}(u) \le -n-1$, we have $\varphi(u \circ_n v) \in U(V)_0^{-n-1}$.

Next we prove that φ is an algebra homomorphism, i.e., $\varphi(u *_n v) = \varphi(u)\varphi(v)$.

Recall that

$$u *_n v = \sum_{m=0}^{n} \sum_{i=0}^{\infty} (-1)^m \binom{m+n}{n} \binom{\Delta_u + n}{i} u_{i-m-n-1} v.$$

We have

$$\begin{aligned}&\varphi(u *_n v)\\ &= \sum_{m=0}^{n} \sum_{i=0}^{\infty} (-1)^m \binom{m+n}{n} \binom{\Delta_u + n}{i} (u_{i-m-n-1} v)(\Delta_u + \Delta_v + m + n - i).\end{aligned}$$

By letting $s = t = 0$ and $N = n$ in Corollary 5.3.2, we have

$$\begin{aligned} J_0(u) J_0(v) &\equiv \sum_{j=0}^{n} \binom{-n-1}{j} J^{(1)}_{n+1,j,-n-1-j}(u,v) \mod U(V)_0^{-n-1} \\ &\equiv \sum_{j=0}^{n} \sum_{i \geq 0} \binom{-n-1}{j} \binom{\Delta_u + n}{i} J_0(u_{-n-1+i-j}) \mod U(V)_0^{-n-1} \\ &\equiv \sum_{j=0}^{n} \sum_{i=0}^{\infty} (-1)^j \binom{n+j}{j} \binom{\Delta_u + n}{i} J_0(u_{i-j-n-1} v) \mod U(V)_0^{-n-1}, \end{aligned}$$

that is, $\varphi(u *_n v) = \varphi(u)\varphi(v)$.

Finally, we want to construct an inverse map for φ. By Lemma 5.3.3, every element of $\dfrac{U(V)_0}{U(V)_0^{-n-1}}$ can be expressed as $J_0(u) + U(V)_0^{-n-1}$ for some $u \in V$. We want to define the map φ^{-1} from $\dfrac{U(V)_0}{U(V)_0^{-n-1}}$ to $A_n(V)$ sending $J_0(u) + U(V)_0^{-n-1}$ to $u + O_n(V)$. Once we prove that this is a well-defined map, it is an inverse for φ. The well-definedness requires that whenever $J_0(u) \in U(V)_0^{-n-1}$, we have $u \in O_n(V)$, i.e., φ^{-1} does not depend on the representatives of an element of $\dfrac{U(V)_0}{U(V)_0^{-n-1}}$. Consider the induced module $\overline{M}(A_n(V))$ constructed in (5.3), where $A_n(V)$ is the regular module of $A_n(V)$. If $J_0(u) \in U(V)_0^{-n-1}$, then $J_0(u)$ will kill the subspace $\overline{M}_n(A_n(V))(n)$, which by Theorem 5.1.7 is isomorphic to $A_n(V)$ itself as $A_n(V)$-modules. Therefore, $J_0(u)v = u *_n v$ for all $v \in V$. In particular, for $v = |0\rangle$, which is the identity element of $A_n(V)$, we have $u *_n |0\rangle = J_0(u)|0\rangle = 0$, which implies that $u \in O_n(V)$. □

Corollary 5.3.5. The Zhu algebra is isomorphic to a subquotient of the universal enveloping algebra,

$$\mathrm{Zhu}(V) = A_0(V) \cong \frac{U(V)_0}{U(V)_0^{-1}}.$$

Recall that there is a surjective Lie algebra homomorphism from $\hat{V}(0)$ to $A_n(V)$, which induces a surjective associative algebra homomorphism from $U(\hat{V}(0))$ to $A_n(V)$. Composing it with the isomorphism (5.6), we can conclude that $U(\hat{V}(0))$ is a dense subalgebra of $U(V)_0$.

Corollary 5.3.6. The subalgebra $U(\hat{V}(0))$ is dense in $U(V)_0$, i.e., $U(\hat{V}(0)) + U(V)_0^{-n} = U(V)_0$ for all $n \geq 0$, hence we have isomorphisms:

$$\frac{U(V)_0}{U(V)_0^{-n-1}} \cong \frac{U(V(0))}{U(V(0))^{-n-1}}.$$

Let $C_2V := \mathrm{span}\{u_{-2}v \mid u, v \in V\}$. A vertex operator algebra V is called C_2-*cofinite* if $\dim \dfrac{V}{C_2V} < \infty$. In [MNT10], the authors proved that if V is C_2-cofinite, then all the subquotients $\dfrac{U(V)_0}{U(V)_0^{-n-1}}$ are finite dimensional. With the isomorphisms between $A_n(V)$ and these subquotients, we easily get the corollary below.

Corollary 5.3.7. If V is a C_2-cofinite vertex operator algebra, then all of its higher level Zhu algebras are finite dimensional.

Chapter 6

Conclusion

Main results

In Chapter 2, we have first shown that non-degenerate invariant bilinear forms exist on truncated current Lie algebras (Lemma 2.1.3) and a good $\mathbb{Z}$-grading on a semi-simple Lie algebra $\mathfrak{g}$ induces a good $\mathbb{Z}$-grading on the truncated current Lie algebra $\mathfrak{g}_p$ (Lemma 2.2.1). Finite W-algebras associated to truncated current Lie algebras were then defined similarly to the semi-simple case (Definition 2.2.8). We have also shown that finite W-algebras in the truncated current versions share some properties of finite W-algebras in the semi-simple case. See, for example, Theorem 2.3.10, Theorem 2.4.2 and Theorem 2.4.8.

In Chapter 3, we have developed an adjusted semi-infinite cohomology theory and also given a characterization of the differential in the adjusted semi-infinite cohomology (Theorem 3.3.9). This adjusted version of semi-infinite cohomology clarifies the definition of affine W-algebras associated to general nilpotent elements given in [KRW03].

In Chapter 4, we have defined affine W-algebras associated to truncated current Lie algebras through (adjusted) semi-infinite cohomology in a uniform way (Theorem 4.3.9).

In Chapter 5, we have studied a general property of higher level Zhu algebras of a vertex operator algebra and proved that they are all isomorphic to subquotients of the universal enveloping algebra (Theorem 5.3.4), hence generalizing a result of I. Frenkel and Y. Zhu [FZ92].

Future research directions

One motivation for this project was the observation that finite W-algebras are the Zhu algebras of affine W-algebras [DSK06]. While higher level Zhu algebras of affine W-algebras are well-defined [vE11], we asked ourselves (thanks to my advisor Michael Lau for this question) what should be those higher level Zhu algebras of affine W-algebras? Is it possible to define them similarly to finite W-algebras?

As shown by Theorem 2.3.10, finite W-algebras associated to truncated current Lie algebras are quantizations of jet schemes of Slodowy slices (in the semi-simple cases), and we know that there are natural maps between jet schemes. On the other hand, there are also natural surjective maps between higher level Zhu algebras. This observation makes us to ask the following question.

Question 1. What is the relationship between W-algebras associated to truncated current Lie algebras and those associated to semi-simple Lie algebras? What is the relationship between finite W-algebras associated to truncated current Lie algebras and higher level Zhu algebras of affine W-algebras associated to semi-simple Lie algebras?

Skryabin equivalence (Theorem 2.4.8) establishes a close relation between the representation theory of truncated current Lie algebras and that of finite W-algebras. Also, the representation theory of truncated current Lie algebras is closely related to that of semi-simple Lie algebras.

Question 2. Study the representation theory of finite and affine W-algebras associated to truncated current Lie algebras and the relationship with that of truncated current Lie algebras.

The theory of adjusted semi-infinite cohomology that we developed in this thesis can potentially apply to quasi-finite $\mathbb{Z}$-graded Lie algebras without any semi-infinite structure, so we have the following natural question.

Question 3. Find an example of quasi-finite $\mathbb{Z}$-graded Lie algebra such that ordinary semi-infinite cohomology does not apply but adjusted semi-infinite cohomology applies.

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