Model stability under a policy shift:
Are DSGE models really structural?

Mémoire présenté
à la Faculté des études supérieures de l'Université Laval
dans le cadre du programme de maîtrise en économique
pour l'obtention du grade de maître ès arts (M.A.)

FACULTÉ DES SCIENCES SOCIALES
UNIVERSITÉ LAVAL
QUÉBEC

2007

©Debbie Gendron, 2007
Résumé

Abstract

This thesis develops a dynamic stochastic general equilibrium model (DSGE) in the line of Ireland (2001), Dib (2003), and Dib, Gammoudi, and Moran (2005). The model is estimated with Canadian time series via Bayesian techniques by combining the likelihood of its state-space representation with prior information and simulating parameter values from their posterior density using the Metropolis-Hastings algorithm. A stability test is then performed on the model by comparing its out-of-sample forecasting ability when estimated on two different samples. The first sample runs from 1981Q1 to 2000Q4, whereas the second starts in 1991Q1 to take into account the inflation targeting regime introduced by the Bank of Canada. Our main finding is that although the parameter estimates related to the monetary policy change significantly following the policy shift, the model’s forecasting ability remains unaffected.
When you've worked on a paper like this over a period of two years, in between classes and work, and full time during the summer, it's easy to forget how you started up, but you'll never forget the people who encouraged and helped you throughout the journey.

The first two people I would like to thank are my thesis supervisors, Stephen Gordon and Kevin Moran, without whose excellent guidance the following would be nonexistent.

To Stephen I owe my understanding of Bayesian econometrics, a branch of econometrics I was ignorant of upon my arrival at Laval University. Stephen's patience and support combined with his ability to render some complex concepts so clear has been of the greatest value. Also, I would like to thank him for his help with Matlab; without his preliminary programs, I would have been quite lost.

To Kevin I owe the discovery of macro models of the type in the following pages. His teachings have encouraged me to develop both criticism and curiosity as to what is out there to help with the understanding and analysis of macroeconomic interactions and policy evaluation. I would also like to thank him for letting me knock on his door whenever I felt like it, for answering all my emails and most trivial questions, and even for encouraging me to present my paper in front of an audience.

To both and to the CIRPEE-Laval, thank you for the financial support that was offered to me, both in scholarships and as a research assistant.

To my family, thank you for encouraging me throughout my studies, and especially Lyna who actually thinks I'm the best!

To my friends, thanks to those who feigned interest in what I was doing and let me ramble on about what was right or wrong at the time. Thanks to those of you who actually are interested and gave me input, asked questions or clarifications that helped me better transmit what I had to say. Particularly, I would like to thank Luc
Acknowledgements

Bissonnette for his help on LATEX.

Also, I would like to thank all the teachers I had but most especially all the macro teachers, Pierre Fortin (UQAM), Victoria Miller (UQAM), Benoît Carmichael (LAVAL) and Kevin Moran (LAVAL), for being so interesting and making me love macroeconomics.

To the reader, thank you for taking the time to read this thesis which represents a significant part of my education. I hope you will enjoy reading it as much as I enjoyed writing and working on it.
Contents

Résumé ii
Abstract iii
Acknowledgements iv
Contents vi
List of Tables viii
List of Figures ix
1 Introduction 1
2 Models and macroeconomics 4
3 A policy régime shift in Canada 6
4 Model spécification 8
  4.1 Households ........................................... 8
  4.2 The final-good-producing firm ......................... 11
  4.3 The intermediate-good-producing firms .............. 11
  4.4 The monetary authority ................................ 13
  4.5 Solving the model ...................................... 14
5 Estimation and data 16
  5.1 Bayesian econometrics: an overview ................... 16
  5.2 Bayesian estimation of a DSGE model ................. 18
    5.2.1 Priors ............................................ 18
    5.2.2 Likelihood ........................................ 20
    5.2.3 The Posterior density ............................. 21
  5.3 Data .................................................. 23
6 Estimation results 24
Contents

7 Forecasting comparison 27
8 Conclusion 34

Bibliography 35

A Symmetric equilibrium equations 39
B Transformed (stationary) system 41
C Steady-state solutions 43
D Linearisation: Blanchard and Kahn 44
E Blanchard and Kahn’s decomposition 47
F Using the Kalman filter for likelihood evaluation 51
G Posterior means and NSE 53
H Forecasting results 54
List of Tables

5.1 Prior distribution of parameters ........................................ 20
6.1 Posterior Means of Parameters ........................................ 26
7.1 Probability of the model’s forecast ................................... 33
G.1 Posterior means’ point estimate ....................................... 53
H.1 Forecasting results using the entire sample period parameter’s estimates 54
H.2 Forecasting results using the subsample period parameter’s estimates 55
# List of Figures

<table>
<thead>
<tr>
<th>Number</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>Canadian Time Series</td>
<td>7</td>
</tr>
<tr>
<td>7.1</td>
<td>Forecasting Output</td>
<td>28</td>
</tr>
<tr>
<td>7.2</td>
<td>Forecasting Real Money Balances</td>
<td>29</td>
</tr>
<tr>
<td>7.3</td>
<td>Forecasting the Interest Rate</td>
<td>30</td>
</tr>
<tr>
<td>7.4</td>
<td>Forecasting the Inflation Rate</td>
<td>31</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

There is growing literature in macroeconomics using dynamic stochastic general equilibrium (DSGE) models for analysis. Based on the general equilibrium framework of the real business cycle (RBC) models, the DSGE models now routinely incorporate market imperfections and nominal rigidities and possess realistic business cycle properties. The DSGE approach is to formulate a model as the result of intertemporal, optimizing behavior on the part of different classes of economic agents. The dynamic properties of the model are thus traceable to behavioral assumptions, leading to a high degree of transparency. In contrast to purely statistical alternatives, DSGE models are less prone to instability because their structure is anchored in microeconomic theory. Consequently, they should be more robust to different policy shifts. For this reason, they are increasingly seen as important policy analysis vehicles.\(^1\)

The question we ask is: Are DSGE models really robust to policy shifts? In particular, are they stable over time? An answer to these questions is necessary if DSGE models are to be used for policy analysis in an ever-changing environment. According to Lucas (1976), the decision from either the monetary or fiscal authority of a country to change a key economic variable in a world where agents are forward-looking and rational must be reflected endogenously through the agent’s optimal behavior. Authors have shown, however, that in some forward-looking models, structural parameters’ estimates may be unstable over time. For instance, Ireland (2001) estimated the parameters of a DSGE model of the American economy over different sample periods and showed that many structural parameter estimates changed over subsamples. In addition, Estrella and Furher (1999) undertake stability tests in both forward-looking and backward-looking

\(^1\)For instance, the Bank of Canada has recently published its technical report describing ToTEM - a sticky-price dynamic stochastic general equilibrium model of the Canadian economy - that has become the new quarterly projection model. [http://www.bankofcanada.ca/fr/res/tr/trlist-f.html](http://www.bankofcanada.ca/fr/res/tr/trlist-f.html)
models and find that some backward-looking models perform better than their forward-looking counterparts as to parameter stability. Taking for granted that parameters in a forward-looking model are invariant could therefore result in very important forecasting or policy evaluation mistakes.

This thesis examines this important question of stability by estimating the structural parameters of a DSGE model with Canadian data, first using a sample of data running through 1981Q1-2000Q4, and then repeating the exercise with a subsample starting in 1991Q1. Afterward, we compare the out-of-sample forecasts of the model estimated on the different samples. We chose 1991Q1 as a breaking date because in 1991, the Bank of Canada started inflation targeting which constitutes an important shift in the practice of monetary policy in Canada.

Estimation of the model's parameters is conducted with a Bayesian approach (e.g. following Smets and Wouters, 2003), and the stability test is conducted by comparing the out-of-sample forecasts of the model estimated under the two different samples with the use of Bayesian model comparison.

Our main finding is that although some parameters significantly change following the policy shift, the model's forecasting ability remains unaffected. The only parameters that do change are those in the monetary policy rule, which indicates that the model has been able to capture the policy shift of 1991. As to the other structural parameters, they remain stable over time. When comparing the forecasting ability of the model estimated over our two samples, neither method overwhelmingly outperforms the other. This allows us to conclude that the model is stable over time.

To our knowledge, this thesis is the first attempt to test for parameter stability in a DSGE model of the Canadian economy. In addition, although Canadian data has been used before to estimate DSGE models (Dib 2003, Dib, Gammoudi and Moran 2005), this thesis reports the first Bayesian estimation of a DSGE model in a Canadian context. The use of a Bayesian approach is very interesting in this context and allows us to explicitly use prior information about parameter values which renders unnecessary calibration of certain parameters and therefore yields a complete description of all parameters' posterior distribution.

The remaining of this thesis is organized as follows: Chapter 2 takes a closer look at the Lucas Critique and the evolution of macroeconomics leading up to the DSGE models. In Chapter 3, we propose evidence of a policy shift in Canada and its importance on the data. Chapter 4 develops the prototypical DSGE model under study. In Chapter 5, we describe the estimation methodology of the model's parameters and the data. In Chapter 6, we expose our estimation results and compare them to other

\[2\] Bordo and Redish, 2005.
studies. We perform the forecasting test and discuss our results in Chapter 7. Finally, Chapter 8 will conclude the thesis.
Chapter 2

Models and macroeconomics

Let us begin with Robert Lucas’ seminal paper: “Econometric Policy Evaluation: A Critique” (1976). In this article, Lucas criticizes 1960s-style macroeconomics in which the economy was modeled as a system of equations based on decision rules. The idea Lucas puts forth is that behind these decision rules lies the behavior of different economic agents, and the behavior of these agents is determined through expectations about the future state of the economy. Estimating the parameters of the decision rules, and then taking them as given in order to conduct policy evaluation can therefore be misleading if agents adjust their behavior to policy changes, thus modifying the underlying parameter values of the equations. Lucas points out that macroeconomic models should be developed at the level of microeconomic decisions and insists on the importance of modeling agents’ behavior.

To answer Lucas’ criticism, a new methodology is developed in the early 80s, particularly by Kydland and Prescott (1982), known as the real business cycle (RBC) methodology. In these models, explicit intertemporal optimization problems are presented and solved, and general equilibrium is the aggregation of optimal behavior in the different sectors of an economy. With this framework, dynamic behavior is traceable to tastes and technologies of the different decision-making agents in the economy and their best expectations regarding the future state of the economy. Thus, the structure of the economy is detailed in a way that is coherent with Lucas’ critique.

However, the canonical RBC model is soon subjected to criticism because it is unable to replicate certain aspects of the observed business cycle. The criticism however is not directed to the modeling framework, but rather toward certain assumptions of

1 For example, Cogley and Nason (1995) show that many RBC models have weak internal propagation mechanisms.
the standard model. For instance, business cycle fluctuations in an RBC model arise from technology shocks only, while many researchers believe that additional shocks are responsible for some of the aggregate fluctuations. In particular, the earlier models are silent about the role of monetary policy in business cycle fluctuations even though there is ample evidence that “Money Matters” and that the short-run stabilization capabilities of the monetary authority can only be modeled in an environment with some type of nominal friction, absent in the RBC model.

The more recent stages of this branch of macroeconomic modeling, known by its acronym DSGE (dynamic stochastic general equilibrium) models build on the basic RBC model by adding features better equipped to replicate business cycle fluctuations and to address monetary policy issues. Many of today’s researchers interested either in policy evaluation or forecasting for example have been using the DSGE framework for their analysis. The most common models incorporate either price or wage rigidities, or both, and an interest-rate rule followed by the monetary authority. Depending on the interests of the researchers, some will work on an open-economy model, while others on a closed-economy model, some will add a government sector, etc. But even the simplest models offer very encouraging results both in their forecasting performance or their ability to match business cycle properties. Consequently, they have become very attractive alternatives to more statistical models as policy analysis tools for both monetary and fiscal authorities.

2See Smets and Wouters (2003) for an example of many different shocks that may be present in an economy.

3See Romer and Romer (1999).

4We refer the reader to Smets and Wouters (2003) and Ratto, Roger, Veld, and Girardi (2005) for treatment of the European area using a DSGE model; to Ireland (2001), Smets and Wouters (2005) and Ambler, Guay, and Phaneuf (2003) for treatment of the US economy; to Dib, Gammoudi, and Moran (2005), Dib (2003) and Amano and Murchison (2005) for examples using Canadian data. These are but a few of the existing literature on DSGE models.
Chapter 3

A policy regime shift in Canada

The Bank of Canada is Canada's central bank and is responsible for the control of monetary policy, with an aim to promote the economic and financial well-being of Canada. Over the past thirty years, there have been different views taken by the Bank as to what economic indicators it should use to justify action on financial markets. From 1975 to 1982, the Bank's immediate target was the short-term interest rate and its intermediate target was M1. Because of the strong demand-elasticity of M1 to the short-term interest rate, the variations in the interest rate necessary to respect the targeted expansion of M1 were too small to prevent large gaps in total spending and inflation compared to their desired trajectories. M1 was thus abandoned as a target in 1982. The following decade seems to have been occupied by the search of a new monetary aggregate to replace M1. The absence of a consensus led the Bank to concentrate on its monetary policy ultimate objective: price stability (Duguay and Poloz, 1994).

Following this decade of uncertainty, an inflation target was jointly agreed upon in 1991 by the Canadian government and the Bank of Canada. The initial goal was to reduce inflation progressively so the inflation target followed a declining schedule: first to 3 per cent, then to 2.5 per cent, finally to 2 per cent to ensure a climate favorable for long-lasting economic growth. By December 1993, inflation had been reduced to 2 per cent. At that time, the government and the Bank agreed to extend the inflation-control target range until the end of 1998, with a range of 1 to 3 per cent. In February 1998, the target range was extended to the end of 2001, in May 2001, it was renewed until the end of 2006, and on November 23 2006, it was again renewed until 2011.

Figure 3.1 depicts the time series of important Canadian macroeconomic variables over the sample period we are interested in (1981-2005). The data is taken from Statistic
Chapter 3. A policy regime shift in Canada

Canada's database (see section 5.3 for a detailed discussion of the series in question). Notice that there seems to be a change in the evolution of the time series around the beginning of inflation targeting in 1991. Output shows a swift decline, followed by a period of stagnation and then growth, while real money balances enter a period of reduced growth, whereas inflation and interest rates are both on a declining path. In addition, there appears to be a decrease in the variability of output. This impression is confirmed by Debs (2001). He shows that there was a structural break in the variability of Canadian production growth during the first quarter of 1991.

The introduction of inflation targeting appears to have lessened uncertainty associated with inflation as credibility of the Bank’s intentions has grown and has stabilized expectations. Together, less uncertainty, and more stable expectations have led to an increase in the length of different contracts both on financial and labor markets.\(^1\)

The introduction of inflation targeting in 1991 therefore constitutes an important shift in the practice of monetary policy in Canada, which may have significantly modified the behavior of different Canadian time series. In this context, it is important to establish whether this shift in monetary policy has had an impact on macroeconomic models. In particular, are DSGE models such as described in the previous section immune to such a shift? Are parameter estimates modified following the 1991 policy change? And if so, are these changes important for the forecasting accuracy of the model? These are the questions we are answering in the following sections.

Figure 3.1: Canadian Time Series

\(^1\)See Longworth (2002) for evidence of lengthened contracts.
Chapter 4

Model specification

In this chapter we formulate the DSGE model estimated in Chapter 5. The model under study is a closed economy model of the Canadian economy similar to the one presented in Dib, Gammoudi and Moran (2005), Dib (2003), and Ireland (2001). The economy is populated by households, final-good-producing firms, intermediate-good-producing firms, and a monetary authority. The final good market is perfectly competitive: each firm produces the same final good and is a price taker for input and output prices. The price of the final good adjusts to clear the market. Final good production is divided between consumption and investment. The extent to which the capital stock can be modified through investment is restricted by capital adjustment costs borne by the households who own the economy’s capital. Final good production uses as inputs an array of intermediate goods. In contrast to the final good’s market, the intermediate goods’ market is monopolistically competitive. Taking as given the demand arising from final-good production, each intermediate-good-producing firm produces a distinct good, under the constraint that rigidities (following Calvo, 1983) affect its ability to change the price of its good. Finally, the monetary authority responds to changes in inflation, output, and money growth by managing a short-term nominal interest rate, similar to Taylor (1993).

4.1 Households

The representative household has preferences defined over consumption \( C_t \), real money balances \( M_t/P_t \), and leisure \( 1-H_t \) (where \( H_t \) represents hours worked) during each period \( t = 0, 1, 2... \) as described by the following expected lifetime utility function:
Chapter 4. Model specification

\[ U_0 = E_0 \sum_{t=0}^{\infty} \beta^t u \left( C_t, \frac{M_t}{P_t}, H_t \right), \]  
(4.1)

where $\beta \in (0, 1)$ is the discount factor. The period utility function is specified as:

\[ u(C_t, \frac{M_t}{P_t}, H_t) = \frac{\gamma z_t}{\gamma - 1} \log \left( \frac{C_t^{\frac{1}{\gamma}}} {\gamma} + b_t^2 \left( \frac{M_t}{P_t} \right)^{\frac{2}{\gamma}} \right) + \eta \log(1 - H_t), \]  
(4.2)

where $\eta$ and $\gamma$ are positive structural parameters. The first governs the disutility of labour (utility of leisure), and the second is related to the intratemporal elasticity between consumption and real money holdings. $z_t$ and $b_t$ are two serially correlated shocks. The preference shock $z_t$, as shown by McCallum and Nelson (1999), resembles a shock to the IS curve in more traditional Keynesian analysis. On the other hand, $b_t$ is interpreted as a shock to money demand. The two shocks evolve according to the following autoregressive processes:

\[ \log(z_t) = \rho_z \log(z_{t-1}) + \varepsilon_{zt}, \]  
(4.3)

\[ \log(b_t) = (1 - \rho_b) \log(b) + \rho_b \log(b_{t-1}) + \varepsilon_{bt}, \]  
(4.4)

with $\rho_z$ and $\rho_b \in (-1, 1)$, and $\varepsilon_{zt}$ and $\varepsilon_{bt}$ are serially uncorrelated innovations that are normally distributed with zero mean and standard deviations $\sigma_z$ and $\sigma_b$, respectively.

The representative household carries $M_{t-1}$ units of nominal money balances, $B_{t-1}$ units of bonds, and $K_t$ units of capital into period $t$. During period $t$, the household receives a lump-sum nominal transfer $T_t$ from the monetary authority as well as dividend payments $D_t$ from the intermediate-good-producing firms. Further, the household supplies labor and capital to the intermediate-good-producing firms, for which it receives total factor payment $R_{kt}K_t + W_tH_t$, where $R_{kt}$ is the nominal rental rate for capital, and $W_t$ is the nominal wage. The household uses some of its funds to purchase the final good at the nominal price $P_t$, which is then divided between consumption and investment. The remainder of available funds are allocated to money holdings $M_t$ and financial bonds $B_t$, which are priced $1/R_t$, where $R_t$ denotes the gross nominal interest rate between $t$ and $t + 1$. The household’s budget constraint is therefore given by:

\[ P_t(C_t + I_t) + M_t + \frac{B_t}{R_t} \leq R_{kt}K_t + W_tH_t + M_{t-1} + B_{t-1} + T_t + D_t. \]  
(4.5)

Investment $I_t$ increases the capital stock, $K_t$, over time according to:

\[ K_{t+1} = (1 - \delta)K_t + I_t - \varphi \left( \frac{K_{t+1}}{K_t} - g \right)^2 K_t, \]  
(4.6)

\[ \]  

The utility function as formulated here is a CES between consumption and real-money balances, and separable with leisure. Another formulation that has recently been growing in popularity is that of habit formation. However, as demonstrated in CEE (2005), habit formation plays a central role only when matching consumption, a dimension of the data that isn't the focus of our analysis.
where $\delta \in (0, 1)$ is a constant capital depreciation rate, $\varphi > 0$ is the capital adjustment cost parameter, and $g$ is the steady state growth rate of the economy to be discussed further below. With this configuration, the cost of changing the capital stock increases with the speed of desired adjustment giving the household an incentive to change capital investment gradually. In addition, total and marginal capital adjustment costs equal zero in the balanced-growth steady state.

The representative household chooses $C_t, M_t, H_t, K_{t+1}, B_t$ in order to maximize expected lifetime utility (4.1). The problem can be written in its recursive form, with its optimal solution satisfying the following Bellman equation:

$$V(K_t, B_{t-1}, M_t; \Omega_t) = \max_{C_t, M_t, H_t, B_t} \left[ u(c_t, M_t, H_t) + E_t \beta V(K_{t+1}, B_t, M_t; \Omega_{t+1}) \right],$$

where $\Omega_t$ is the information available at time $t$. The decision making process of households must respect conditions (4.5) and (4.6).

Solving for the representative household’s optimal behavior yields the following first order conditions:

$$C_t : \frac{z_t C_t^{\gamma-1}}{C_t^{\gamma-1} + b_t^{1} (M_t/P_t)^{\gamma-1}} = \lambda_t,$$

$$M_t : \frac{z_t b_t^{1} (M_t/P_t)^{\gamma-1}}{C_t^{\gamma-1} + b_t^{1} (M_t/P_t)^{\gamma-1}} = \lambda_t - \beta E_t \left( \frac{P_t \lambda_{t+1}}{P_{t+1}} \right),$$

$$H_t : \frac{\eta}{1 - H_t} = \lambda_t \frac{W_t}{P_t},$$

$$K_{t+1} : \varphi \left( \frac{K_{t+1}}{K_t} - g \right) + 1 = \beta E_t \left[ \frac{\lambda_{t+1}}{\lambda_t} \left( \frac{R_{t+1}}{P_{t+1}} \right) + 1 - \delta + \varphi \left( \frac{K_{t+2}}{K_{t+1}} - g \right) \right],$$

$$B_t : \frac{1}{R_t} = \beta E_t \left( \frac{P_t \lambda_{t+1}}{P_{t+1} \lambda_t} \right),$$

where $\lambda$ is the Lagrangian multiplier associated to the budget constraint (4.5).

As in Ireland (1997), and Dib, Gammoudi, and Moran (2005), equations (4.8), (4.9), and (4.12) together imply the following standard money-demand equation:

$$\log(M_t/P_t) \approx \log(C_t) - \gamma \log(r_t) + \log(b_t),$$

where $r_t = R_t - 1$ denotes the net, nominal interest rate between $t$ and $t + 1$, $\gamma$ is the money-interest elasticity, and $b_t$ represents a serially correlated shock to money demand.
Chapter 4. Model specification

4.2 The final-good-producing firm

The final good $Y_t$ is produced from an array of intermediate goods $Y_{jt}$, with $j \in [0,1]$, according to the following aggregation function:

$$ Y_t = \left( \int_0^1 Y_{jt}^{\theta-1} \, dj \right)^{\theta}, \theta > 1, $$

(4.14)

where $\theta$ represents the elasticity of substitution between intermediate goods.

Each final-good-producing firm maximises profits taking as given the market price $P_t$ of the final good and the prices $p_{jt}$ of the input goods. It will therefore choose the quantity of intermediate goods $Y_{jt}$ that will maximise profits as follows:

$$ \max_{\{Y_{jt}\}_{j=0}^{1}} \left[ P_t Y_t - \int_0^1 p_{jt} Y_{jt} \, dj \right], $$

(4.15)

subject to the production function in (4.14). The first-order condition for $Y_{jt}$ implies the following input demand of the final-good-producing firm for intermediate good $j$:

$$ Y_{jt} = \left( \frac{p_{jt}}{P_t} \right)^{-\theta} Y_t. $$

(4.16)

This input demand function will serve as the economy-wide demand for good $j$ in the intermediate firms’ sector below.

Since the final good market is perfectly competitive, aggregate profits in the sector must be zero. Imposing this zero profit condition to (4.15) and using (4.16) yields the following definition of the final good’s price:

$$ P_t = \left( \int_0^1 p_{jt}^{1-\theta} \, dj \right)^{-\frac{1}{1-\theta}}. $$

(4.17)

4.3 The intermediate-good-producing firms

The intermediate-good-producing firm $j$ rents $K_{jt}$ units of capital, and hires $H_{jt}$ units of labor in order to produce $Y_{jt}$ units of output according to the following constant-returns-to-scale technology:

$$ Y_{jt} \leq A_t K_{jt}^\alpha \left( g^* H_{jt} \right)^{1-\alpha}, \alpha \in (0,1), $$

(4.18)
where $A_t$ is a stochastic (stationary) aggregate technology shock common to all intermediate-good-producing firms, whereas $g^t$ is the non-stochastic growth rate associated to technological progress.\(^2\) $A_t$ is assumed to follow the autoregressive process:

$$\log(A_t) = (1 - \rho_A)\log(A) + \rho_A\log(A_{t-1}) + \epsilon_{At}, \quad (4.19)$$

with $\rho_A \in (-1, 1)$, and $\epsilon_{At}$ is a serially uncorrelated innovation to $A_t$, normally distributed with zero mean and standard deviation $\sigma_A$.

We solve the intermediate-good-producing firm’s problem using a two step method. The first step consists in a minimisation of costs subject to the production function (4.18) where $mc_t$ is the Lagrangian multiplier reflecting the firm’s marginal cost of producing one additional unit of good:

$$\min_{\{K_{jt}, H_{jt}\}} \frac{R_{kt}K_{jt}}{P_t} + \frac{W_tH_{jt}}{P_t} + mc_t[Y_{jt} - A_tK_{jt}^\alpha \left(g^tH_{jt}\right)^{1-\alpha}]. \quad (4.20)$$

The first order conditions for this problem are:

$$K_{jt}: \frac{R_{kt}}{P_t} = \alpha mc_t \frac{Y_{jt}}{K_{jt}}, \quad (4.21)$$

$$H_{jt}: \frac{W_t}{P_t} = (1 - \alpha) mc_t \frac{Y_{jt}}{H_{jt}}. \quad (4.22)$$

By replacing the first order conditions for capital and labor in the minimisation problem, we may write the total costs as:

$$TC_t = mc_tY_{jt}. \quad (4.23)$$

The second step consists of maximising profits (the product of price and quantity less the firm’s total costs). It is well known that money is neutral in a monopolistically competitive framework unless some nominal frictions are added to the model. Here, nominal rigidity is introduced following Calvo (1983). According to this specification, firms are not allowed to re-optimise their price unless they receive a random signal. Specifically, with probability $1 - \phi$ the firm receives the signal and chooses a new price, whereas with probability $\phi$ the firm cannot re-optimise, but may index its price to the steady-state inflation rate, $\pi$.\(^3\)

\(^2\) We could have written the production function as: $Y_{jt} \leq K_{jt}^{\alpha}(A_tH_{jt})^{1-\alpha}$, where $A_t$ would be one single process with both trend and shock component, but for manipulation purposes, it was more convenient to express the idea as in the text.

\(^3\) This type of indexation for firms that do not receive the signal to re-optimise follows Yun (1996). In CEE (2005) firms that do not re-optimise index their prices to lagged inflation. Amano and Murchison (2005) combine both methods by applying partial dynamic indexation.
When firm \( j \) receives the signal to re-optimise, it chooses a price \( \tilde{p}_{jt} \) as well as a contingency plan for \( H_{jt+k} \) and \( K_{jt+k} \) in order to maximise its discounted, expected total profits for the expected length of time during which the price \( \tilde{p}_{jt} \) will remain in effect. The profit maximisation problem is the following:

\[
\max_{\{\tilde{p}_{jt}\}} E_0 \left[ \sum_{k=0}^{\infty} (\beta \phi)^k \lambda_{t+k} \left( \pi^k \tilde{p}_{jt} / P_{t+k} - mc_t \right) Y_{jt+k} \right].
\] (4.24)

The probability that the price \( \tilde{p}_{jt} \) remains in effect for \( t + k \) periods is reflected by the inclusion of \( \phi^k \).

Profit maximisation is undertaken subject to the demand for good \( j \) in (4.16). After some simple algebra, the first-order condition for this optimisation problem can be written as:

\[
\hat{\tilde{p}}_{jt} = \frac{\theta}{\theta - 1} \frac{E_t \sum_{k=0}^{\infty} (\beta \phi \pi^{-\theta})^k \lambda_{t+k} Y_{t+k} mc_{t+k} P_{t+k}^\theta}{E_t \sum_{k=0}^{\infty} (\beta \phi \pi^{1-\theta})^k \lambda_{t+k} Y_{t+k}^{-1} P_{t+k}^{\theta-1}}.
\] (4.25)

Because of the symmetry in the demand for their good (4.16), all firms that receive the signal to re-optimsie choose the same price \( \tilde{p}_{jt} \), which will henceforth be denoted \( \tilde{p}_t \). Considering condition (4.17) for the final good's price, and the fact that at the aggregate level a fraction \( 1 - \phi \) of intermediate-good-producing firms re-optimsie, the aggregate price index evolves according to:

\[
P_t^{1-\theta} = \phi (\pi P_{t-1})^{1-\theta} + (1 - \phi) \tilde{p}_t^{1-\theta}.
\] (4.26)

An interesting feature of this model is obtained from the first-order approximation of equation (4.25) and (4.26). Together, they yield a New-Keynesian Phillips curve relating the present period's inflation rate to the expectation of future rates as well as today's marginal costs, an indicator of the strength of economic activity:

\[
\hat{\pi}_t = \beta \hat{\pi}_{t+1} + \frac{(1 - \phi)(1 - \beta \phi)}{\phi} \hat{mc}_t,
\] (4.27)

where circumflexes denote the percentage deviation of the variables from their deterministic steady-state.

### 4.4 The monetary authority

We assume that the Bank of Canada responds to deviations of inflation \( \pi_t = P_t / P_{t-1} \), output \( Y_t \), and money growth \( \mu_t = M_t / M_{t-1} \) by managing the nominal interest-rate \( R_t \),
(Taylor, 1993). The interest-rate rule is given by:

$$\log(R_t/R) = \theta_x \log(\pi_t/\pi) + \theta_y \log(Y_t/Y) + \theta_\mu \log(\mu_t/\mu) + \log(v_t), \quad (4.28)$$

where $R$, $\pi$, $Y$, and $\mu$ are the steady-state values of $R_t$, $\pi_t$, $Y_t$, and $\mu_t$ respectively. Further, $v_t$ is a monetary policy shock that evolves according to the following autoregressive process:

$$\log(v_t) = \rho_v \log(v_{t-1}) + \epsilon_{vt}, \quad (4.29)$$

where $\rho_v \in [0,1]$ and $\epsilon_{vt}$ is a serially uncorrelated innovation to $v_t$ with zero mean and standard deviation $\sigma_v$.

### 4.5 Solving the model

The model is solved using a symmetric equilibrium solution. In a symmetric equilibrium, all intermediate-good-producing firms make identical decisions. Let $r_{kt} = R_{kt}/P_t$, $W_t = W_t/P_t$, $m_t = M_t/P_t$, be the real rental rate of capital, the real wage, and real money-balances respectively. The economy is in equilibrium when the allocations $(Y_t, c_t, M_t, H_t, K_t)_{t=1}^{\infty}$, prices and co-state variables $(W_t, r_{kt}, R_t, \pi_t, \lambda_t, mc_t)_{t=1}^{\infty}$, and the different shocks are such that households and both types of firms optimise, the monetary policy rule is satisfied, and the following market-clearing conditions are satisfied:

- $K_t = \int_0^1 K_{jt}dj$,
- $H_t = \int_0^1 H_{jt}dj$,
- $m_t = \bar{m}_t$,
- $B_t = 0$,
- $Y_t = C_t + I_t$.

This model admits a unique deterministic steady state in which consumption, output, investment, and real-money balances grow at the same growth rate $g$, while hours worked remain constant. Since a closed formed solution to this model does not exist, the model is loglinearized around that deterministic steady state. The model’s approximate dynamics around this steady state can then be written in a state-space representation using Blanchard and Kahn’s (1980) decomposition.\(^4\) For trended variable, let lowercase variables correspond to their uppercase version divided by the growth rate $(y_t \equiv Y_t/g)$, and let circumflexes denote the deviations of these variables from their steady state, so that $\hat{y}_t = \log(y_t/y_{ss})$. The model’s following state-space representation is obtained:

\(^4\)See appendix E for the system decomposition
where \( \hat{s}_t \) is a vector of state variables that includes the predetermined and exogenous variables, \( \hat{d}_t \) is a vector of control variables, and \( \epsilon_t \) contains the random innovations.\(^5\)

The elements of the matrices \( \Phi_1, \Phi_2, \Phi_3 \) depend on the model’s structural parameters, \( \Phi.\)\(^6\)

The state-space representation can be used to estimate the underlying parameters of the model via Bayesian econometric techniques involving optimisation through Monte-Carlo Markov-Chain (MCMC) sampling of the posterior density function.

---

\[^5\] \( \hat{s}_t = \begin{bmatrix} \hat{k}_{t-1} & \hat{m}_{t-1} & \hat{A}_t & \hat{\delta}_t & \hat{\varepsilon}_t \end{bmatrix}, \hat{d}_t = \begin{bmatrix} \hat{y}_t & \hat{R}_t & \hat{\xi}_t & \hat{\epsilon}_t \end{bmatrix}, \hat{\epsilon}_t = \begin{bmatrix} \hat{\varepsilon}_A & \hat{\varepsilon}_B & \hat{\varepsilon}_C & \hat{\varepsilon}_D \end{bmatrix}. \)

\[^6\] \( \Phi = \begin{bmatrix} \rho_A & \rho_B & \rho_C & \rho_D & \sigma_A & \sigma_B & \sigma_C & \sigma_D & \beta & \delta & \eta & \theta & \alpha & \phi & \pi & \gamma \end{bmatrix}. \)
Chapter 5

Estimation and data

In this section, we describe the econometric approach and data used for estimating the model's 23 structural parameters. But first, let us introduce Bayesian econometrics in its simplicity. Readers already familiar with Bayesian econometrics may skip to section 5.2 without loss of continuity.

5.1 Bayesian econometrics: an overview

This section is meant for the readers not familiar with the Bayesian approach in econometrics. It is a basic overview of its theoretical foundations. For readers interested in a more detailed explanation, we refer you to Koop (2003) and his many references.

Baye's rule, which is central to Bayesian econometrics, is derived from the rules of probability. Consider two random variables A and B; then the rules of probability imply:

\[ p(A, B) = p(A|B)p(B) = p(B|A)p(A), \]  

(5.1)

where \( p(A, B) \) is the joint density, \( p(A|B) \) is the conditional density and \( p(A) \) is the marginal density. Rearranging the terms yields Baye's rule:

\[ p(A|B) = \frac{p(B|A)p(A)}{p(B)}. \]  

(5.2)

In an econometric model, we are interested in the joint probability of a parameter vector \( \Phi \), and the data \( d \), \( p(\Phi, d) \), which can be written as in (5.1), and then rearranged using Baye's rule as:

\[ p(\Phi|d) = \frac{p(d|\Phi)p(\Phi)}{p(d)}. \]  

(5.3)
The fundamental interest is with \( p(\Phi|d) \), the posterior density. Bayesians directly address the question: 'Given the data, what do we know about the parameters?' Since \( p(d) \) does not involve \( \Phi \), we may rewrite Baye's rule as:

\[
p(\Phi|d) \propto p(d|\Phi)p(\Phi)
\]

(5.4)

where \( p(\Phi) \) is the prior density summarizing information the researcher has about the model’s parameters prior to estimation, and \( p(d|\Phi) \) is the likelihood function which is the density of the data conditional on the parameters; also referred to as the data generating process. Therefore, \( p(\Phi|d) \), the posterior density, is a summary of all the information we have about the parameters. It can be seen as an updating rule where the data allows us to update our prior views about the parameters.

Even though \( p(\Phi|d) \) incorporates all the information available about \( \Phi \), we are often interested in certain characteristics of the distribution, for example, its mean or variance. In general, let \( g(\Phi) \) be a function of interest, its mathematical expectation can be written as:

\[
E[g(\Phi|d)] = \int g(\Phi)p(\Phi|d)d\Phi.
\]

(5.5)

An analytical calculation of the integrals involved in (5.5) is possible only in a few simple cases. For more complex models, the use of a posterior simulator is necessary. There exists different posterior simulators. However, they are all extensions of the laws of large numbers or central limit theorem.

**Theorem 1 : Monte Carlo Integration**

Let \( \Phi^S \) for \( s = 1, \ldots, S \) be a random sample from \( p(\Phi|d) \), and define

\[
\hat{g}^S = \frac{1}{S} \sum_{s=1}^{S} g(\Phi^S),
\]

(5.6)

then \( \hat{g}^S \) converges to \( E[g(\Phi|d)] \) as \( S \) goes to infinity.

So if it is possible to compute a random sample from the posterior, (5.6) allows us to approximate \( E[g(\Phi|d)] \) by averaging the function of interest evaluated at the random sample. However, only if \( S \) were infinite would the approximation error go to zero. A way of reporting the approximation error is by calculating Geweke's (1992) numerical standard error. If the sequence \( \Phi^S \) is simulated from independent draws from a distribution with \( \sigma^2 \) variance, the mean's \( (S^{-1} \sum \Phi^S) \) variance is simply \( S^{-1}\sigma^2 \). In the case where the sequence of draws is not independent, the mean's variance of a sample
Chapter 5. Estimation and data

where observations $g(\Phi^S)$ are correlated is $S^{-1}S_g(0)$, where $S_g(0)$ is the spectral density of $g(\Phi^S)$ evaluated at 0.

In the next section, we explain how the Bayesian approach is applied to our model’s specific state-space representation.

5.2 Bayesian estimation of a DSGE model

The model’s solution was previously written in its state-space form, which will be used to estimate the 23 structural parameters (i.e. recall equations (4.30) and (4.31)):

$$\hat{s}_{t+1} = \Phi_1 \hat{s}_t + \Phi_2 \epsilon_{t+1},$$

$$\hat{d}_t = \Phi_3 \hat{s}_t.$$  \hfill (5.7)

The joint density we are interested in is that of the parameters in $\Phi_1$, $\Phi_2$, and $\Phi_3$ and the data, $\hat{d}_t$. Let $\Phi$ be a vector containing all parameters of the model, the posterior density of the parameters can then be written as showed in the previous section using Baye’s rule:

$$p(\Phi|\hat{d}_t) \propto p(\hat{d}_t|\Phi)p(\Phi)$$  \hfill (5.9)

where $p(\Phi)$ is the prior density, $p(\hat{d}_t|\Phi)$ is the likelihood function, and $p(\Phi|\hat{d}_t)$ is the posterior density.

5.2.1 Priors

Table 5.1 gives an overview of our assumptions regarding the prior distribution of the 23 estimated structural parameters. All variances of the shocks are assumed to be distributed as an inverted gamma distribution (to guarantee a positive variance) as in Smets and Wouters (2003). The other priors reflect the information available from calibration literature, similar DSGE model estimations, or micro-founded research. The autoregressive parameters in the technology, preference, money-demand, and monetary shocks, as well as the discount rate $\beta$, capital’s share in the production function $\alpha$, the depreciation of capital $\delta$, and the Calvo parameter $\phi$ are assumed to follow a Beta distribution which covers the range between 0 and 1. The first three have prior means of 0.8 since the associated shocks are believed to be persistent, but we let the standard deviation be 0.2 which is a bit less constraining than in Smets and Wouters (2003); the
autoregressive parameter of the monetary policy shock is believed to be less persistent so its prior mean is set at 0.5 with standard deviation 0.2. The prior means for $\beta$, $\alpha$, and $\delta$ are set to 0.97, 0.36, and 0.025 respectively, values commonly used in the literature. We allow $\beta$’s standard deviation to be 0.2, letting it cover a wider range, while $\delta$’s is set to 0.1, and $\alpha$’s to 0.01. Notice that the standard deviation on $\alpha$ is much more restrictive. In previous iterations, we set it less restrictively, and the estimation led to unbelievably high values for $\alpha$; constraining it allows our results to stay in line with the literature. The mean for the length of price contracts $\phi$ is set to 0.5 with standard deviation of 0.3; which allows us to fit Bils and Klenow’s (2004) value (5.5 months), as well as Amirault et al.’s (2005) who show that price flexibility has been growing in Canada. All the other parameters are assumed to follow truncated normal distributions, mostly truncated at 0, or 1 for the growth rate and the elasticity of substitution between different intermediate goods. The monetary policy parameters’ means are set close to the estimated values of Taylor (1993) and Ireland (2001) ($\rho_y = 1.5$, $\rho_y = 0.5$, and $\rho_M = 0.5$), but we let the standard deviations of each cover a wider range (set at 0.5). The steady-state inflation rate is set to 1.01 following Ireland (2001), with standard deviation equal to 0.01. The mean of $\eta$ (the weight of leisure in the utility function), is set to 1.35, implying that households spend about one third of their time working, with standard deviation 0.1. The mean for the elasticity of substitution between the different intermediate goods is 6, which implies an average mark-up of price over marginal cost equal to Rotemberg and Woodford’s (1992) benchmark 20 per cent, with standard deviation equal to 1. The mean of $\chi$’s (the capital adjustment cost parameter) is set to 15 with standard deviation 5 covering the values calibrated in Ireland (2001), and Dib (2003). The mean for the parameter $b$, determining the steady-state ratio of real-balances to consumption, is set to 0.5 with standard deviation 0.2, matching the steady-state consumption velocity of money in the model to the average consumption velocity of M2 in the Canadian data from 1976 to 2000 (see Dib, 2003). The mean of the interest-rate elasticity to money demand ($\gamma$) is set to 0.05 with standard deviation 0.03; this follows Mulligan and Martin (2000) who estimate the interest-rate elasticity to money demand and show that when interest-rates are low, $\gamma$ is also low. The growth rate $g$’s mean is set at 1.005 (representing an annual average growth rate of the economy of 2 per cent) with standard deviation 0.002. $A$ is a scale parameter with mean set to 3000 and standard deviation 500.

1The persistence of the monetary policy shock is often estimated to be less persistent than the other shocks, see Ireland (2001).
Table 5.1: Prior distribution of parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Prior Distribution</th>
<th>Mean</th>
<th>Standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_A$</td>
<td>inverted gamma</td>
<td>0.0141</td>
<td>9.0000e-004</td>
</tr>
<tr>
<td>$\sigma_b$</td>
<td>inverted gamma</td>
<td>0.0141</td>
<td>9.0000e-004</td>
</tr>
<tr>
<td>$\sigma_x$</td>
<td>inverted gamma</td>
<td>0.0141</td>
<td>9.0000e-004</td>
</tr>
<tr>
<td>$\sigma_v$</td>
<td>inverted gamma</td>
<td>0.0141</td>
<td>9.0000e-004</td>
</tr>
<tr>
<td>$\rho_A$</td>
<td>Beta</td>
<td>0.8</td>
<td>0.2</td>
</tr>
<tr>
<td>$\rho_b$</td>
<td>Beta</td>
<td>0.8</td>
<td>0.2</td>
</tr>
<tr>
<td>$\rho_x$</td>
<td>Beta</td>
<td>0.8</td>
<td>0.2</td>
</tr>
<tr>
<td>$\rho_v$</td>
<td>Beta</td>
<td>0.5</td>
<td>0.2</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Beta</td>
<td>0.97</td>
<td>0.02</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Beta</td>
<td>0.36</td>
<td>0.01</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Beta</td>
<td>0.025</td>
<td>0.01</td>
</tr>
<tr>
<td>$\phi$</td>
<td>Beta</td>
<td>0.5</td>
<td>0.3</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Truncated Normal</td>
<td>1.35</td>
<td>0.1</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Truncated Normal</td>
<td>0.05</td>
<td>0.03</td>
</tr>
<tr>
<td>$b$</td>
<td>Truncated Normal</td>
<td>0.5</td>
<td>0.2</td>
</tr>
<tr>
<td>$\chi$</td>
<td>Truncated Normal</td>
<td>15</td>
<td>5</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Truncated Normal</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>$g$</td>
<td>Truncated Normal</td>
<td>1.005</td>
<td>0.002</td>
</tr>
<tr>
<td>$A$</td>
<td>Truncated Normal</td>
<td>3000</td>
<td>500</td>
</tr>
<tr>
<td>$\rho_x$</td>
<td>Truncated Normal</td>
<td>1.5</td>
<td>0.5</td>
</tr>
<tr>
<td>$\rho_y$</td>
<td>Truncated Normal</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>$\rho_\mu$</td>
<td>Truncated Normal</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>$\pi$</td>
<td>Truncated Normal</td>
<td>1.01</td>
<td>0.01</td>
</tr>
</tbody>
</table>

5.2.2 Likelihood

We follow Ireland (1999) in treating the capital stock as a latent variable. With latent variables, we can exploit the recursive nature of the model and apply the Kalman Filter to evaluate the likelihood function $p(\hat{d}_t|\Phi)$. The log likelihood for $\hat{d}_t$, knowing that $\hat{d}_t \sim N(\hat{d}_{t|t-1}, \Phi V_{t|t-1}\Phi')$, is written:

$$lnL = -\frac{NT}{2} ln(2\pi) - \frac{1}{2} \sum_{t=1}^{T} ln|\Phi V_t \Phi'| - \frac{1}{2} \sum_{t=1}^{T} [(\hat{d}_t - \hat{d}_{t|t-1})'(\Phi V_t \Phi')(\hat{d}_t - \hat{d}_{t|t-1})],$$  (5.10)

where $\hat{d}_{t|t-1}$ is $\hat{d}_t$'s mean conditional on all information available at time $t - 1$. For a detailed explanation of the filter, we refer the reader to the notes in Appendix F which
are based on Hamilton (1994, ch.13).

Classical econometricians will simply use the Kalman filter to construct the sample log likelihood and then maximize it numerically with respect to the unknown parameters of interest. However, since our sample data is relatively small, the behavior of a numerical algorithm can be very unstable. To overcome this problem, many researchers calibrate certain parameters so as to insure convergence of the likelihood. Instead, we use a Bayesian approach by adding prior information to make sure the behavior of the parameters is standard. This method is the same used by Smets and Wouters (2003). We therefore need a posterior simulator to recover the posterior density of the parameters since analytical calculation is impossible. The next section completes our estimation procedure by introducing the Metropolis-Hastings algorithm within a Gibbs sampler used to simulate the posterior density of the parameters.

5.2.3 The Posterior density

To simulate from the posterior density, we use the Metropolis-Hastings algorithm within a Gibbs sampler.

The Gibbs sampler is the simplest Monte Carlo Markov Chain (MCMC) technique. For each parameter $\Phi_j$ in a parameter vector $\Phi$, if a model is complicated, the marginal distribution $p(\Phi_j|d)$ is not standard. However, the conditional distribution $p(\Phi_j|\Phi_{j-1},d)$ may be, where $\Phi_{j-1}$ represents all the model's parameters excluding $\Phi_j$. If we can simulate a candidate from the conditional $p(\Phi_j|\Phi_{j-1},d)$, we can generate a random sequence with the following:

\[
\begin{align*}
\Phi_1^i & \sim p(\Phi_1|\Phi_2^{i-1}, \Phi_3^{i-1}, ..., \Phi_{j-1}^{i-1}, d) \quad \text{one iteration} \\
\Phi_2^i & \sim p(\Phi_2|\Phi_1^{i-1}, \Phi_3^{i-1}, ..., \Phi_{j-1}^{i-1}, d) \\
\Phi_3^i & \sim p(\Phi_3|\Phi_1^{i-1}, \Phi_2^{i-1}, ..., \Phi_{j-1}^{i-1}, d) \\
& \vdots \\
\Phi_j^i & \sim p(\Phi_j|\Phi_1^{i-1}, \Phi_2^{i-1}, ..., \Phi_{j-1}^{i-1}, d) \\
\end{align*}
\]

The Markov chain described above converges to its stable distribution $p(\Phi|d)$.

If the conditional distribution $p(\Phi_j|\Phi_{j-1},d)$ is not standard (as in our case), we may use the Metropolis-Hastings algorithm to recover it.

The Metropolis-Hastings algorithm developed by Metropolis and al. (1953) and Hastings (1970) can be used to simulate draws in non-standard distributions. It is a

\footnote{See Geman and Geman (1984), and Gordon and Bélanger (1996), for a more detailed explanation.}
Monte Carlo simulation-based technique which uses Markov Chain properties to assure
cconvergence of the simulated density; thus it is referred to as a MCMC technique. For
more intuition as to its theoretical foundations, we refer the interested reader to Chib
and Greenberg (1994).

To simulate a new draw in the chain, we use a known density, henceforth referred to
as the candidate generating density (CGD). Then, a probabilistic rule is used to decide
whether or not the new candidate is accepted.

Let the CGD be written as \( q(\Phi_j, \Phi'_j) \). If the last value of the chain is \( \Phi_j \), the
candidate \( \Phi'_j \) is simulated from the distribution of the CGD. The latter candidate is
then accepted with probability \( \alpha(\Phi_j, \Phi'_j) \), where:

\[
\alpha(\Phi_j, \Phi'_j) = \min \left\{ \frac{p(\Phi'_j|\tilde{d}_j)q(\Phi'_j, \Phi_j)}{p(\Phi_j|\tilde{d}_j)q(\Phi_j, \Phi'_j)}, 1 \right\}
\]

Once the new candidate \( \Phi'_j \) is simulated, (5.12) is used to calculate the probability
that it will be accepted. A simple way of applying this rule is to simulate a realisation
\( U \) from a uniform distribution \((0,1)\); the candidate is accepted if \( U < \alpha \). Thus, we
may summarize the algorithm to simulate realisation \( \Phi'_j \) of the Markov chain with the
following steps:

1. Given the previous value \( \Phi^{i-1}_j \), simulate a candidate \( \Phi'_j \) from the CDG \( q(\Phi^{i-1}_j, \Phi'_j) \)
2. Calculate \( \alpha(\Phi_j, \Phi'_j) \)
3. Simulate \( U \) from a Uniform distribution \((0,1)\). If \( U < \alpha \), fix \( \Phi'_j \) to \( \Phi_j \); or else, fix
\( \Phi^{i}_j \) to \( \Phi^{i-1}_j \).
4. Repeat

It can be proved (Chib and Greenberg, 1994) that the Markov chain described by
this algorithm is ergodic and converges to \( p(\Phi_j|\tilde{d}_i) \). Then, it is just a matter of simple
arithmetics to calculate point estimate of the mean or variance of the parameters using
Theorem 1 in section 5.1.

A last explanation is needed here as to the choice of a CDG. We use the one suggested
by Metropolis and al. (1953) where \( q(\Phi_j, \Phi'_j) = f(\Phi'_j - \Phi_j) \), where \( f(.) \) is a known
density. In this case, the candidate is simulated from a random walk:

\[
\Phi'_j = \Phi_j + z,
\]
where \( z \sim f(z) \). With the random walk CDG, candidate draws are taken in random directions from the current point. If the innovations of the random walk come from a symmetric distribution, note that \( f(z) = f(-z) \), then \( q(\Phi_j, \Phi'_j) = f(\Phi_j - \Phi'_j) = q(\Phi'_j, \Phi_j) \). If that is the case, the expression \( q(\Phi'_j, \Phi_j)/q(\Phi_j, \Phi'_j) \) in (5.12) disappears and the probability of accepting the new candidate is simply:

\[
\alpha(\Phi_j, \Phi'_j) = \min \left\{ \frac{p(\Phi'_j | d_t)}{p(\Phi_j | d_t)}, 1 \right\}
\]

(5.14)

In other words, if the targeted density \( p(\Phi_j | d_t) \) is higher with \( \Phi'_j \) than it is with \( \Phi_j \), the previous iteration’s value, then we accept the candidate. However, if the value of the density lessens of say \( \delta \), then it is only accepted with probability \((1 - \delta)\).

The form of the CDG is determined by the choice of density for \( z \). The multivariate Normal is the most common and convenient choice. The mean of the Normal is determined by \( \Phi_j \) in 5.13 and the researcher must choose the covariance matrix to insure that the acceptance probability is neither too high nor too low. To find the covariance matrix, one can simply experiment with different values until a reasonable acceptance probability is obtained.\(^3\)

### 5.3 Data

We estimate the model’s parameters using a subset of the control variables \( d_t \) in (5.8) for which data is available. Since the model is driven by four stochastic processes, we estimate it using data for four series to prevent singularity in the determination of the likelihood. The data are taken from CANSIM, Statistics Canada’s database. The series include output, real money balances, a short-term interest-rate, and inflation. Output is measured by real, final domestic demand. Real money balances are measured by dividing M2 (currency and all chequable notices and personal term deposits) by the GDP deflator. The short-term interest-rate is the three-month Treasury Bill rate. Finally, inflation is the gross rate of increase in the GDP deflator. Further, output and real money balances are expressed in per capita terms using the population aged 16 to 64.\(^4\) All data are logged and rendered quarterly (by taking a three month average when necessary) before estimation.

\(^3\)There is no general rule as to the ideal acceptance probability. However, if it is roughly 0.5 one is unlikely to go wrong. (Koop, 2003, 97-98)

\(^4\)The series used are V1992078 for output, V37128 for M2, V122531 for the interest-rate, V1997756 for the GDP deflator, and V2091037 for population which can be found at www.statcan.ca
Chapter 6

Estimation results

We proceeded to several preliminary replication rounds in order to find the convenient
covariance matrix for the random walk CDG. Once convergence was obtained we then
proceeded to calculate the posterior mean for each parameter under the Monte Carlo
Integration theorem 5.6. This procedure was followed for both of our sample periods.
Table 6.1 presents Bayesian posterior means of the model's 23 parameters estimated on
Geweke's (1992) numerical standard error (NSE) was also calculated and in the entire
sample (subsample), the NSE associated to 12 (15) parameter estimates is under 1 per
cent of the absolute value of the parameter's posterior mean, associated to 20 (22) is
under 5 per cent, and associated to 23 is under 17 per cent (11 per cent).¹

Focusing first on the parameters of the policy rule, the estimates of \( \pi \), 1.0182 for
the entire sample and 1.0107 for the subsample, translate into annualized, steady-state
inflation rates of 7.5 per cent and 4.3 per cent respectively. Further, the interest rate
response to inflation, measured by \( \rho_\pi \), is much larger at 1.3926, when estimated with
our subsample, versus 0.7130 when estimated with the entire sample. These estimates
imply that the Bank of Canada has been more aggressive in fighting inflation during
our subsample period, which accords well with our priors and the introduction of the
find similar results for the before-and-after Volcker period at the Federal Reserve in the
United States. Further, the interest rate response to variations in money growth, \( \rho_\mu \), is
also slightly more important in our subsample, another indication of the increased ag-gressiveness of the Bank to inflationary pressures. On the other hand, in both cases, the
interest rate response to variations in output, \( \rho_y \), is practically zero, a result consistent

¹In appendix II is a table with both posterior mean estimates as well as NSE values.
with the literature (see Ireland, 2001, Smets and Wouters, 2003, and Dib, Gammoudi, and Moran, 2005).

Interestingly, all the other parameters seem to remain stable over time, at values resembling those commonly calibrated or estimated in DSGE literature. The discount rate $\beta$ is estimated in the entire sample (subsample) at 0.9877 (0.9884), a value close to the ones used in the calibration literature. The depreciation of capital $\delta$ is slightly higher than what is normally used in calibration (0.025) at 0.0335 (0.0330). The weight on leisure in the representative household’s utility function $\eta$ is 1.3105 (1.3376) which implies that the household spends about one third of its time working; again a commonly calibrated value. Capital’s share in the intermediate good’s production is close to 0.36, another commonly used value. The elasticity of substitution between the different intermediate goods $\theta$ is 6.8217 (6.5425), which implies an average mark-up of price over marginal cost equal to Rotemberg and Woodford’s (1992) benchmark 20 per cent. The estimate for $b$, the parameter determining the steady-state ratio of real balances to consumption is 0.5205 (0.5320); while that of $\gamma$, the elasticity of substitution between consumption and real-money balances is 0.0613 (0.0701), similar to the estimated values of Dib, Gammoudi, and Moran (2005) for the Canadian economy. The estimates of $\phi$, 0.7005 (0.6792), imply that firms, on average, re-optimize their prices about once every 9 months. The preference, money-demand, and technological shocks are all very persistent, whereas the monetary policy shock is much less persistent. Again, this is consistent with findings in Dib, Gammoudi, and Moran (2005).

The fact that there is no significant change to key parameters, apart from the changes in the monetary policy rule, seems to indicate that the model is stable over time. The variations of the estimated parameters in the policy rule simply capture the fact that the monetary policy has changed in 1991. The model seems stable, but how is it to be formally judged? Chapter 7 discusses how we answer this question.
## Table 6.1: Posterior Means of Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimated with the entire sample</th>
<th>Estimated with the subsample</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.9877</td>
<td>0.9884</td>
</tr>
<tr>
<td>$\rho_A$</td>
<td>0.9999</td>
<td>0.9999</td>
</tr>
<tr>
<td>$\sigma_A$</td>
<td>0.0150</td>
<td>0.0134</td>
</tr>
<tr>
<td>$\rho_b$</td>
<td>0.9968</td>
<td>0.9747</td>
</tr>
<tr>
<td>$\sigma_b$</td>
<td>0.0131</td>
<td>0.0131</td>
</tr>
<tr>
<td>$\rho_z$</td>
<td>0.8560</td>
<td>0.7658</td>
</tr>
<tr>
<td>$\sigma_z$</td>
<td>0.0181</td>
<td>0.0150</td>
</tr>
<tr>
<td>$\rho_v$</td>
<td>0.2883</td>
<td>0.3732</td>
</tr>
<tr>
<td>$\sigma_v$</td>
<td>0.0125</td>
<td>0.0141</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.3630</td>
<td>0.3604</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.0335</td>
<td>0.0330</td>
</tr>
<tr>
<td>$\phi$</td>
<td>0.7008</td>
<td>0.6792</td>
</tr>
<tr>
<td>$\eta$</td>
<td>1.3105</td>
<td>1.3376</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.0613</td>
<td>0.0701</td>
</tr>
<tr>
<td>$b$</td>
<td>0.5205</td>
<td>0.5320</td>
</tr>
<tr>
<td>$\chi$</td>
<td>21.556</td>
<td>19.616</td>
</tr>
<tr>
<td>$\theta$</td>
<td>6.8217</td>
<td>6.5425</td>
</tr>
<tr>
<td>$g$</td>
<td>1.0048</td>
<td>1.0047</td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>3131.1</td>
<td>3023.1</td>
</tr>
<tr>
<td>$\rho_\pi$</td>
<td>0.7130</td>
<td>1.3926</td>
</tr>
<tr>
<td>$\rho_y$</td>
<td>0.0015</td>
<td>0.0026</td>
</tr>
<tr>
<td>$\rho_\mu$</td>
<td>1.2057</td>
<td>1.5289</td>
</tr>
<tr>
<td>$\pi$</td>
<td>1.0182</td>
<td>1.0107</td>
</tr>
</tbody>
</table>
Chapter 7

Forecasting comparison

As we mentioned in the previous section, most parameters have remained stable through the monetary policy shift of 1991, although the parameters in the interest rate rule seem to have changed. One way of establishing if the model is stable over time is by comparing its out-of-sample forecasts when estimated over the two different sample periods. This also permits us to evaluate the importance of the variation in parameters on the forecasting ability of the model. One-step-ahead forecasts are evaluated using the predictive density:

\[ p(\hat{d}_{t+1}|\hat{d}_t) = \int p(\hat{d}_{t+1}|\hat{d}_t, \Phi)p(\Phi|\hat{d}_t)d\Phi, \]  

(7.1)

with the optimal mean satisfying:

\[ E[\hat{d}_{t+1}|\hat{d}_t] = \int p(\hat{d}_{t+1}|\hat{d}_t)d\hat{d}_{t+1}, \]  

(7.2)

which can be calculated through our state-space representation by averaging over the forecast of each draw \( \Phi^i \). So, for each of the replications of \( \Phi \) kept from our MCMC posterior simulator, calculate \( \hat{d}_{t+1} \) from the observation equation of the state-space representation:

\[ \hat{d}_{t+1} = P(\Phi)s_t, \]  

(7.3)

then, average over all \( \hat{d}_{t+1} \) to have a one-step-ahead forecast.

Figures 7.1 to 7.4 show for both samples the one-step-ahead forecasts associated with the four time series used to estimate the model. In each graph, there are two panels, the upper panel gives a view of the entire period (1981Q1-2005Q1), while the lower panel enhances the latter part of the upper panel. Note that the data used for estimation runs from 1981Q1 to 2000Q4 (entire sample) and from 1991Q1 to 2000Q4 (subsample), so we are comparing one-step-ahead forecasts with the actual available
data starting with 2001Q1 up to 2005Q1. Also, note that for each new observation added, the model is re-estimated.

Figure 7.1: Forecasting Output

---

Subsample forecasts
• Entire sample forecasts
— Actual data

---

Subsample forecasts
• Entire sample forecasts
— Actual data
Figure 7.2: Forecasting Real Money Balances

log real money balances

Subsample forecasts
Entire sample forecasts
Actual data


Figure 7.3: Forecasting the Interest Rate

Annualized interest rate

- Subsample forecasts
- Entire sample forecasts
- Actual data
Figure 7.4: Forecasting the Inflation Rate

Annualized inflation rate

- Subsample forecasts
- Entire sample forecasts
- Actual data


Simply by glancing at the graphs, we can see that both sample period estimates render similar forecasts.\(^1\)

A more robust way of comparing the forecasting ability of the model under the two different samples is by using Bayesian forecasting model comparison. Let the model estimated on the entire sample period be Model A, and let Model B be the one estimated on the subsample, and let \( p(A|\hat{d}_t) \) and \( p(B|\hat{d}_t) \) be the probabilities associated to each model before the forecasting exercise. Conditional on \( \hat{d}_t \), it is possible to derive the predictive density \( p(\hat{d}_{t+1}|A, \hat{d}_t) \) and \( p(\hat{d}_{t+1}|B, \hat{d}_t) \). Once \( \hat{d}_{t+1} \) is observed, the information set becomes \( D_{t+1} = (\hat{d}_{t+1}, \hat{d}_t) \), and the probability of model A conditional on \( \hat{d}_{t+1} \) is:

\[
p(A|\hat{d}_{t+1}) = \frac{p(\hat{d}_{t+1}|A, \hat{d}_t)p(A|\hat{d}_t)}{p(\hat{d}_{t+1}|A, \hat{d}_t)p(A|\hat{d}_t) + p(\hat{d}_{t+1}|B, \hat{d}_t)p(B|\hat{d}_t)},
\]

where \( p(A|\hat{d}_t) \) and \( p(B|\hat{d}_t) \) are our priors on each model. We did the exercise with 0.5 priors on each model. Then, for the next period, we have:

\[
p(A|\hat{d}_{t+2}) = \frac{p(\hat{d}_{t+2}|A, \hat{d}_{t+1})p(A|\hat{d}_{t+1})}{p(\hat{d}_{t+2}|A, \hat{d}_{t+1})p(A|\hat{d}_{t+1}) + p(\hat{d}_{t+2}|B, \hat{d}_{t+1})p(B|\hat{d}_{t+1})},
\]

so our first posterior probability \( p(A|\hat{d}_{t+1}) \) becomes the prior for our next, and so on.

Table 7.1 presents model probabilities for Model A and Model B on our entire forecast horizon. Notice that the probability associated to the last period is much higher in the subsample so it indicates that it would be the preferred model. However, depending on the date of the last observation added, we might prefer the model estimated on the entire sample period (see the probability associated with period 2002Q4 or 2003Q1 for example). Neither model outperforms the other definitively, which supports the conclusion that the model is in fact stable over time.

\(^1\)In appendix I, you will find tables presenting the forecasting values for each period using both the entire sample and subsample estimates with the associated standard deviation in brackets.
Table 7.1: Probability of the model’s forecast

<table>
<thead>
<tr>
<th>Quarter forecasted</th>
<th>Model A (Entire sample)</th>
<th>Model B (Subsample)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2001Q1</td>
<td>0.4339</td>
<td>0.5661</td>
</tr>
<tr>
<td>2001Q2</td>
<td>0.2811</td>
<td>0.7189</td>
</tr>
<tr>
<td>2001Q3</td>
<td>0.2177</td>
<td>0.7823</td>
</tr>
<tr>
<td>2001Q4</td>
<td>0.6366</td>
<td>0.3634</td>
</tr>
<tr>
<td>2002Q1</td>
<td>0.5796</td>
<td>0.4204</td>
</tr>
<tr>
<td>2002Q2</td>
<td>0.7441</td>
<td>0.2559</td>
</tr>
<tr>
<td>2002Q3</td>
<td>0.5677</td>
<td>0.4323</td>
</tr>
<tr>
<td>2002Q4</td>
<td>0.6440</td>
<td>0.3560</td>
</tr>
<tr>
<td>2003Q1</td>
<td>0.7086</td>
<td>0.2914</td>
</tr>
<tr>
<td>2003Q2</td>
<td>0.2997</td>
<td>0.7003</td>
</tr>
<tr>
<td>2003Q3</td>
<td>0.3965</td>
<td>0.6035</td>
</tr>
<tr>
<td>2003Q4</td>
<td>0.2659</td>
<td>0.7341</td>
</tr>
<tr>
<td>2004Q1</td>
<td>0.4399</td>
<td>0.5601</td>
</tr>
<tr>
<td>2004Q2</td>
<td>0.5545</td>
<td>0.4455</td>
</tr>
<tr>
<td>2004Q3</td>
<td>0.4239</td>
<td>0.5761</td>
</tr>
<tr>
<td>2004Q4</td>
<td>0.2963</td>
<td>0.7037</td>
</tr>
<tr>
<td>2005Q1</td>
<td>0.2756</td>
<td>0.7244</td>
</tr>
</tbody>
</table>
Chapter 8

Conclusion

This paper has developed and estimated a closed economy DSGE model of the Canadian economy using a Bayesian approach. We provide estimates of its structural parameters over two subsamples separated by the introduction of inflation targeting in 1991 by the Bank of Canada, in order to establish whether the model is stable over time. Our estimates suggest that the model is able to capture the monetary policy shift. In fact, the estimated parameters of the Taylor rule followed by the monetary policy appear to change significantly in 1991 suggesting a more aggressive reaction from the Bank to inflationary pressures. By contrast, the other parameter estimates are unchanged from one sample to the other. We also compare the one-step-ahead forecasting ability of the model estimated over the two subsamples and find that neither outperforms the other when compared to the actual data. Finally, to provide a more complete analysis, we undergo a Bayesian model comparison test based on the model’s predictive densities and find that the preferred model depends on the date of the last observation added.

Overall, our results support the view that the estimated DSGE model is stable, which is very promising for further research in developing and estimating DSGE models for forecasting and policy analysis purposes. Considering that the closed economy hypothesis in the context of the Canadian economy might not be accurate, a next challenge for researchers would be to adapt the model to an open-economy to provide a complete stability analysis.
Bibliography


Appendix A

Symmetric equilibrium equations

Define:  \( r_{kt} = R_{kt}/P_t \)

Market clearing conditions:  \( K_t = \int_0^1 K_{j,t} \, dt, \quad H_t = \int_0^1 H_{j,t} \, dt, \quad Y_t = \int_0^1 Y_{j,t} \, dt \)

\[
\frac{z_t}{C_t^{\gamma}} \left( \frac{1}{C_t^{\gamma-1} + b_t^1 (M_t/P_t)^{\gamma-1}} \right) = \lambda_t
\]

\[
\frac{z_t b_t^1 (M_t/P_t)^{\gamma-1}}{C_t^{\gamma-1} + b_t^1 (M_t/P_t)^{\gamma-1}} = \lambda_t - \beta E_t \left( \frac{\lambda_{t+1}}{\pi_{t+1}} \right)
\]

\[
\frac{\eta}{1 - H_t} = \lambda_t \left( W_t/P_t \right)
\]

\[
\varphi \left( \frac{K_{t+1}}{K_t} - g \right) + 1 = \beta E_t \left[ \frac{\lambda_{t+1}}{\lambda_t} \left( r_{kt+1} + 1 - \delta + \varphi \left( \frac{K_{t+2}}{K_{t+1}} - g \right) \frac{K_{t+2}}{K_{t+1}} - \frac{\varphi}{2} \left( \frac{K_{t+2}}{K_{t+1}} - g \right)^2 \right) \right]
\]

\[
\frac{1}{R_t} = \beta E_t \left( \frac{\lambda_{t+1}}{\lambda_t \pi_{t+1}} \right)
\]

\[
r_{kt} = \alpha mc_t \frac{Y_t}{K_t}
\]

\[
(W_t/P_t) = (1 - \alpha) mc_t \frac{Y_t}{H_t}
\]

\[
\bar{p}_t = \frac{\theta \pi_t}{\theta - 1} E_t \sum_{k=0}^{\infty} \left( \beta \phi \pi_{t-k} \right)^k \lambda_{t+k} mc_{t+k} Y_{t+k} \left( \prod_{s=1}^{k} \pi_{t-s}^{\theta} \right)
\]

\[
Y_t = A_t K_t^\alpha (g' H_t)^{1-\alpha}
\]

\[
K_{t+1} = (1 - \delta) K_t + Y_t - C_t - \varphi/2 \left( \frac{K_{t+1}}{K_t} - g \right)^2 K_t
\]
\[ \pi_t^{1-\theta} = (1 - \phi) \tilde{\pi}_t^{1-\theta} + \phi \pi \]  
(A.11)

\[ \mu_t = \frac{(M_t/P_t) \pi_t}{(M_{t-1}/P_{t-1})} \]  
(A.12)

\[ \log(R_t/R) = \vartheta_z \log(\pi_t/\pi) + \vartheta_y \log(Y_t/Y) + \vartheta_{\mu} \log(\mu_t/\mu) + \log(\nu_t), \]  
(A.13)

\[ \log(z_t) = \rho_z \log(z_{t-1}) + \epsilon_{zt} \]  
(A.14)

\[ \log(b_t) = (1 - \rho_b) \log(b) + \rho_b \log(b_{t-1}) + \epsilon_{bt} \]  
(A.15)

\[ \log(A_t) = (1 - \rho_A) \log(A) + \rho_A \log(A_{t-1}) + \epsilon_{At} \]  
(A.16)

\[ \log(\nu_t) = \rho_{\nu} \log(\nu_{t-1}) + \epsilon_{\nu t}, \]  
(A.17)
Appendix B

Transformed (stationary) system

Since certain variables have an upward time trend, we define them in terms of stationary (detrended) variables:

\[ c_t = C_t / g^t \]
\[ k_t = K_t / g^t \]
\[ i_t = I_t / g^t \]
\[ y_t = Y_t / g^t \]
\[ (M_t / P_t) / g^t = m_t \]
\[ h_t = H_t \]
\[ (W_t / P_t) / g^t = w_t \]
\[ \lambda_t = \lambda_t g^t \]

So as to write:

\[
\frac{z_t c_t^{\frac{1}{\eta}}} {c_t^{\frac{1}{\eta}} + b_t^{\frac{1}{\eta}} m_t^{\frac{1}{\eta}}} = \lambda_t \quad (B.1)
\]

\[
\frac{z_t b_t^{\frac{1}{\eta}} m_t^{\frac{1}{\eta}}} {c_t^{\frac{1}{\eta}} + b_t^{\frac{1}{\eta}} m_t^{\frac{1}{\eta}}} = \lambda_t - \beta E_t \left( \frac{\lambda_{t+1}} {\pi_{t+1}} \right) \quad (B.2)
\]

\[
(1/g^t)^{1-h_t} = \lambda_t w_t \quad (B.3)
\]

\[
\varphi \left( \frac{g k_{t+1}} {k_t} - g \right) + 1 = \beta E_t \left[ \frac{\lambda_{t+1}} {\lambda_t} \left( r_{k_{t+1}} + 1 - \delta + \varphi \left( \frac{g k_{t+2}} {k_{t+1}} - g \right) \frac{g k_{t+2}} {k_{t+1}} - \frac{\varphi}{2} \left( \frac{g k_{t+2}} {k_{t+1}} - g \right)^2 \right) \right] \quad (B.4)
\]

\[
\frac{1}{R_t} = \beta E_t \left( \frac{\lambda_{t+1}} {\lambda_t \pi_{t+1}} \right) \quad (B.5)
\]

\[
r_{kt} = \alpha m c_t \frac{y_t}{k_t} \quad (B.6)
\]
Appendix B. Transformed (stationary) system

\[ w_t = (1 - \alpha)mc_i \frac{y_t}{h_t} \]  
(B.7)

\[ \widetilde{p}_t = \frac{\theta \pi_t}{\theta - 1} \frac{E_t \sum_{k=0}^{\infty} (\beta \phi \pi^{-\theta})^k \lambda_{k+1} mc_t k y_t + k \sum_{s=1}^{k} \pi_{t+s}^\theta}{E_t \sum_{k=0}^{\infty} (\beta \phi \pi^{1-\theta})^k \lambda_{k+1} y_t + k \sum_{s=1}^{k} \pi_{t+s}^{\theta-1}} \]  
(B.8)

\[ y_t = A_t k_t^\alpha (h_t)^{1-\alpha} \]  
(B.9)

\[ g_{k+1} = (1 - \delta)k_t + c_t - \frac{\varphi}{2} (g_{k+1}^2 - g^2)_{k_t} \]  
(B.10)

\[ \pi_t^{1-\theta} = (1 - \phi)\bar{p}_t^{1-\theta} + \phi \pi \]  
(B.11)

\[ \mu_t = \frac{gm_t \pi_t}{m_t^{-1}} \]  
(B.12)

\[ \log(R_t/R) = \theta_n \log(\pi_t/\pi) + \theta_y \log(y_t/y) + \theta_\mu \log(\mu_t/\mu) + \log(v_t), \]  
(B.13)

\[ \log(z_t) = \rho_z \log(z_{t-1}) + \epsilon_z \]  
(B.14)

\[ \log(b_t) = (1 - \rho_b) \log(b) + \rho_b \log(b_{t-1}) + \epsilon_b \]  
(B.15)

\[ \log(A_t) = (1 - \rho_A) \log(A) + \rho_A \log(A_{t-1}) + \epsilon_A \]  
(B.16)

\[ \log(v_t) = \rho_v \log(v_{t-1}) + \epsilon_v, \]  
(B.17)
Appendix C

Steady-state solutions

The following steady-state solutions are derived from equations B1-B17.

\[
\begin{align*}
\mu &= g \pi \\
R &= \frac{\pi}{\beta} \\
mc &= \frac{\theta - 1}{\theta} \\
r_k &= \frac{1}{\beta} - 1 + \delta \\
k &= \frac{\alpha mc}{y} \\
c &= 1 + (1 - \delta - g) \frac{k}{y} \\
\lambda c &= \frac{1}{1 + b \left( \frac{\pi}{\beta} \right)^{\gamma - 1}} \\
\lambda m &= \lambda c \pi^{\gamma} (\pi - \beta)^{-\gamma b} \\
h &= \frac{\lambda c (1 - \alpha) mc}{\eta (c/y)} \\
y &= (\frac{k}{y})^{\gamma - \alpha} A^{\frac{1}{1 - \alpha}} h \\
w &= (1 - \alpha) mc (y/h)
\end{align*}
\]

We now know \( h/(1 - h) = x \), so \( h = x/(x + 1) \), we know \( x \) so we know \( h \), from which we may find ... \( c = c/y \times y \), \( k = k/y \times y \), \( \lambda = \lambda c/e \), \( m = \lambda m/\lambda \), \( w = (1 - \alpha)mc \times y/h \).
Appendix D

Linearisation : Blanchard and Kahn

Static equations:
\[ A f_t = Bd_t + C x_t \]  
(D.1)

Dynamic equations:
\[ D d_{t+1} + E f_{t+1} = F d_t + G f_t + H x_t \]  
(D.2)

with A, B, C, D, F, G, H being matrices, f is a vector of flow variables, d is a vector of dynamic variables and x is a vector of shocks.

Static Equations
\[ \dot y_t + (\alpha - 1)\dot h_t = \alpha \dot k_t + \dot A_t \]  
(D.3)

\[ \dot m_t - \dot \pi_t = \dot m_t - \dot m_{t-1} \]  
(D.4)

\[ \dot c_t = \gamma \dot \lambda_t + \lambda m_t - \dot m_t \]  
(D.5)

\[ - [1 + \lambda c(\gamma - 1)]\dot c_t = \gamma \dot \lambda_t + \lambda m_t - \dot m_t \]  
(D.6)

Dynamic Equations
\[ \beta \pi_{t+1} = \pi_{t-1} - \frac{(1 - \beta \phi)(1 - \phi)}{\phi} \dot m_t \]  
(D.11)

\[ (\varphi g(\beta(1-\delta) - (1 + \beta g)))\dot k_{t+1} + (\beta(r_k + 1 - \delta))\dot \lambda_{t+1} + \beta r_k \dot \lambda_{k+1} + \beta \varphi g_k \dot y_{t+1} - \beta \varphi g_k \dot c_{t+1} = - \varphi g \dot k_t + \dot \lambda_t \]  
(D.12)
Appendix D. Linearisation: Blanchard and Kahn

\[ \hat{k}_{t+1}g = (1 - \delta)k\hat{k}_t + \hat{y}_t y - \hat{c}_t c \]  
\[ \hat{\lambda}_{t+1} - \hat{\pi}_{t+1} = \hat{\lambda}_t - \hat{R}_t \]  
\[ \hat{m}_t = \hat{m}_t \]

The previous equations may be written in their matrix form as in D.1 and D.2 with the following elements in each lines/columns:

\[
A = \begin{pmatrix}
    \hat{y}_t & \hat{R}_t & \hat{\pi}_t & \hat{\pi}_t & \hat{\pi}_t & \hat{\pi}_t & \hat{\pi}_t \\
    1 & 0 & 0 & 0 & 0 & 0 & \alpha - 1 & 0 \\
    0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\
    1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & -1 - (\gamma - 1)\lambda c & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & -\lambda c(\gamma^{-1}) & 0 & 0 & 0 & 0 \\
    1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\
    -\theta_y & 1 & 0 & 0 & -\theta_y & 0 & 0 & -\theta_y \\
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
    \hat{k}_t & \hat{m}_{t-1} & \hat{\lambda}_t & \hat{\pi}_t & \hat{\pi}_t & \hat{\pi}_t & \hat{\pi}_t \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & -1 & 0 & 0 & 1 & 0 & 0 \\
    0 & 0 & -1 & 0 & 0 & 0 & 0 \\
    0 & 0 & \gamma & 0 & \lambda m(\gamma - 1)\frac{\pi - \beta}{\pi} & 0 & 0 \\
    0 & 0 & 1 & 0 & \lambda m\frac{\gamma - 1}{\gamma}\frac{\pi - \beta}{\pi} + 1 & 0 & 0 \\
    0 & 0 & 0 & -1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
C = \begin{pmatrix}
    \hat{A}_t & \hat{b}_t & \hat{v}_t & \hat{z}_t \\
    1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & \lambda m\frac{\pi - \beta}{\pi} & 0 & -\gamma \\
    0 & \lambda m\frac{\pi - \beta}{\pi} - \frac{1}{\gamma} & 0 & -1 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
Appendix D. Linearisation: Blanchard and Kahn

\[ D = \begin{pmatrix}
    \hat{k}_{t+1} & \hat{m}_t & \hat{\lambda}_{t+1} & \hat{m}_c_{t+1} & \hat{m}_{t+1} \\
    0 & 0 & 0 & 0 & 0 \\
    g\varphi(\beta(1-\delta) - (1+g\beta)) & 0 & \beta(r_k + 1 - \delta) & 0 & 0 \\
    gk & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 \\
    0 & 1 & 0 & 0 & 0 
\end{pmatrix} \]

\[ E = \begin{pmatrix}
    \hat{y}_{t+1} & \hat{R}_{t+1} & \hat{r}_{kt+1} & \hat{c}_{t+1} & \hat{\pi}_{t+1} & \hat{w}_{t+1} & \hat{h}_{t+1} & \hat{\mu}_{t+1} \\
    0 & 0 & 0 & 0 & \beta & 0 & 0 & 0 \\
    \frac{g\varphi g}{k} & 0 & \beta r_k & -\frac{g\varphi g}{k} & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 
\end{pmatrix} \]

\[ F = \begin{pmatrix}
    \hat{k}_t & \hat{m}_{t-1} & \hat{\lambda}_t & \hat{m}_c_t & \hat{m}_t \\
    0 & 0 & 0 & \frac{(1-\beta\phi)(1-\phi)}{\phi} & 0 \\
    -\varphi g & 0 & 1 & 0 & 0 \\
    (1 - \delta)k & 0 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 0 & 1 
\end{pmatrix} \]

\[ G = \begin{pmatrix}
    \hat{y}_t & \hat{R}_t & \hat{r}_{kt} & \hat{c}_t & \hat{\pi}_t & \hat{w}_t & \hat{h}_t & \hat{\mu}_t \\
    0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    y & 0 & 0 & -e & 0 & 0 & 0 & 0 \\
    0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 
\end{pmatrix} \]
Appendix E

Blanchard and Kahn’s decomposition

Static equation:
\[ Af_t = Bd_t + Cx_t \]  \hspace{1cm} (E.1)

Dynamic equation:
\[ Dd_{t+1} + Ef_{t+1} = Fd_t + Gf_t + Hx_t \]  \hspace{1cm} (E.2)

Shocks:
\[ x_{t+1} = \rho x_t + \xi_{t+1}, \xi = N(0, \sigma) \]  \hspace{1cm} (E.3)

First step: isolate \( f \) in the static equation
\[ f_t = A^{-1}Bd_t + A^{-1}Cx_t \]  \hspace{1cm} (E.4)

One period ahead:
\[ f_{t+1} = A^{-1}Bd_{t+1} + A^{-1}Cx_{t+1} \]  \hspace{1cm} (E.5)

Replace \( f_t \) and \( f_{t+1} \) in dynamic equation:
\[ Dd_{t+1} + E[A^{-1}Bd_{t+1} + A^{-1}Cx_{t+1}] = Fd_t + G[A^{-1}Bd_t + A^{-1}Cx_t] \]  \hspace{1cm} (E.6)

Remember: \( E_t(x_{t+1}) = \rho x_t \), that we may replace in previous equation:
\[ Dd_{t+1} + E[A^{-1}Bd_{t+1} + A^{-1}C\rho x_t] = Fd_t + G[A^{-1}Bd_t + A^{-1}Cx_t] \]  \hspace{1cm} (E.7)

Put same variables together:
\[ [D + EA^{-1}B]d_{t+1} = [F + GA^{-1}B]d_t + [GA^{-1}C - EA^{-1}C\rho]x_t \]  \hspace{1cm} (E.8)
Appendix E. Blanchard and Kahn's decomposition

Isolate $d_{t+1}$

$$d_{t+1} = [D + EA^{-1}B]^{-1}[F + GA^{-1}B]d_t + [D + EA^{-1}B]^{-1}[GA^{-1}C - EA^{-1}C\rho]x_t \quad (E.9)$$

Compactly:

$$d_{t+1} = Kd_t + Lx_t \quad (E.10)$$

Diagonalize $K$: $K = M^{-1}NM$, with $M$ being eigenvectors and $N$ being eigenvalues, $N$

$$= \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix} \text{ with } N_1 < 1 \text{ and } N_2 > 1 \text{ being vectors.}$$

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \text{ with } M_{ij} \text{ being vectors. Also write } L = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}$$

So $d_{t+1} = M^{-1}NMd_t + Lx_t$

Premultiply by $M$: $Md_{t+1} = NMd_t + MLx_t$

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}d_{t+1} = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}d_t + \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}x_t$$

Define: $Q_1 = M_{11}L_1 + M_{12}L_2$ and $Q_2 = M_{21}L_1 + M_{22}L_2$

Also define $d_t = [k_t \lambda_t]'$, where $k_t$ are predetermined variables and $\lambda_t$ are not predetermined. Therefore, associate $N_1$ to $k_t$ and $N_2$ to $\lambda_t$.

Define: $\hat{k}_t = M_{11}k_t + M_{12}\lambda_t$ and $\hat{\lambda}_t = M_{21}k_t + M_{22}\lambda_t$

So: $\hat{k}_{t+1} = N_1\hat{k}_t + Q_1x_t$

$$\hat{\lambda}_{t+1} = N_2\hat{\lambda}_t + Q_2x_t$$

We must cancel out the effect of unstable eigenvalues $N_2 > 1$, so $\lim_{N \to \infty} \hat{\lambda}_t = 0$

Knowing that $\hat{\lambda}_{t+1} = N_2\hat{\lambda}_t + Q_2x_t$, we may isolate $\hat{\lambda}_t$:

$$\hat{\lambda}_t = N_2^{-1}\hat{\lambda}_{t+1} - N_2^{-1}Q_2x_t \quad (E.11)$$

Recursively:

$$\hat{\lambda}_t = N_2^{-1}(N_2^{-1}\hat{\lambda}_{t+2} - N_2^{-1}Q_2x_{t+1}) - N_2^{-1}Q_2x_t \quad (E.12)$$

By continuous recursion:

$$\hat{\lambda}_t = N_2^{-(n+1)}\hat{\lambda}_{t+n} - \sum_{s=0}^{n} N_2^{-(s+1)}Q_2x_{t+s} \quad (E.13)$$
Appendix E. Blanchard and Kahn's décomposition

Remember that: \[ \lim_{N \to \infty} N_2^{-(n+1)} \hat{\lambda}_{t+n+1} = 0, \text{ and } E(x_{t+s}) = \rho^s x_t \]

\[ \hat{\lambda}_t = -\sum_{s=0}^{n} N_2^{-(s+1)} Q_2 \rho^s x_t \]  
(E.14)

Or more compactly:

\[ \hat{\lambda}_t = u x_t \]  
(E.15)

Let’s go back to the equation: \[ \hat{\lambda}_t = M_{21} k_t + M_{22} \lambda_t \] Isolate \( \lambda_t \)

\[ \lambda_t = M_{22}^{-1} \hat{\lambda}_t - M_{22}^{-1} M_{21} k_t \]  
(E.16)

Replace \( \hat{\lambda}_t \):

\[ \lambda_t = M_{22}^{-1} u x_t - M_{22}^{-1} M_{21} k_t \]  
(E.17)

Write more compactly:

\[ \lambda_t = s_2 x_t + s_1 k_t \]  
(E.18)

We may now replace \( \lambda_t \):

\[ M_{11} k_{t+1} + M_{12} \lambda_{t+1} = N_1 M_{11} k_t + Q_1 x_t \]  
(E.19)

Replace \( \lambda_{t+1} \):

\[ M_{11} k_{t+1} + M_{12} [s_2 x_{t+1} + s_1 k_{t+1}] = N_1 M_{11} k_t + Q_1 x_t \]  
(E.20)

Isolate \( k_{t+1} \):

\[ k_{t+1} = [M_{11} + M_{12} s_1]^{-1} N_1 M_{11} k_t + [M_{11} + M_{12} s_1]^{-1} Q_1 - M_{12} s_2 \rho |x_t \]  
(E.21)

Compactly:

\[ k_{t+1} = s_3 k_t + s_4 x_t \]  
(E.22)

We may go back to the static equation and replace \( \lambda \):

\[ f_t = A^{-1} B d_t + A^{-1} C x_t \]  
(E.23)

\[ f_t = A^{-1} B \begin{pmatrix} k_t \\ \lambda_t \end{pmatrix} + A^{-1} C x_t \]  
(E.24)

\[ f_t = A^{-1} B \begin{pmatrix} k_t \\ s_1 k_t + s_2 x_t \end{pmatrix} + A^{-1} C x_t \]  
(E.25)

\[ f_t = A^{-1} B \begin{pmatrix} I \\ s_1 \end{pmatrix} k_t + [A^{-1} C + A^{-1} B \begin{pmatrix} 0 \\ s_2 \end{pmatrix}] x_t \]  
(E.26)

More compactly:

\[ f_t = s_5 k_t + s_6 x_t \]  
(E.27)
Matrix form of state equation:

\[
\begin{pmatrix}
    k_{t+1} \\
    x_{t+1}
\end{pmatrix} = \begin{pmatrix}
    s_3 & s_4 \\
    0 & \rho
\end{pmatrix} \begin{pmatrix}
    k_t \\
    x_t
\end{pmatrix} + \begin{pmatrix}
    0 \\
    I
\end{pmatrix} \xi_{t+1}
\]

Matrix form of observation equation:

\[
\begin{pmatrix}
    f_t \\
    \lambda_t
\end{pmatrix} = \begin{pmatrix}
    s_5 & s_6 \\
    s_1 & s_2
\end{pmatrix} \begin{pmatrix}
    k_t \\
    x_t
\end{pmatrix}
\]

State-space form:

Observation equation:

\[d_t = \Phi_3 s_t\] (E.28)

State equation:

\[s_{t+1} = \Phi_1 s_t + \Phi_2 \epsilon_{t+1}\] (E.29)
Appendix F

Using the Kalman filter for likelihood evaluation

State-space representation of the model:

\[ s_{t+1} = \Phi_1 s_t + \Phi_2 \epsilon_{t+1} \]  \hspace{1cm} (F.1)
\[ d_t = \Phi_3 s_t \]  \hspace{1cm} (F.2)

where \( \epsilon_t \sim N(0, \Sigma) \).

The Kalman filter is used to calculate linear least square forecasts of the state vector on the basis of data observed at time \( t \). It is used to generate projections recursively from \( \hat{s}_{1|0}, \hat{s}_{2|1}, \) to \( \hat{s}_{T|T-1} \).

Associated to each projection is the mean square error (MSE):

\[ P_{t+1|t} = E[(s_{t+1} - \hat{s}_{t+1|t})(s_{t+1} - \hat{s}_{t+1|t})'] \]  \hspace{1cm} (F.3)

where \( \hat{s}_{t+1|t} = E[s_{t+1}|d_t] \) is a linear projection of \( s_t \) on \( d_t \) and a constant.

To start the recursion, we start with \( \hat{s}_{1|0} \) which is a forecast of \( s_1 \) based on no observation, and is simply the unconditional mean of \( s_1 \).

\[ \hat{s}_{1|0} = E(s_1) \]  \hspace{1cm} (F.4)
\[ P_{1|0} = E[(s_1 - E(s_1))(s_1 - E(s_1))'] \]  \hspace{1cm} (F.5)

From the state equation, the unconditional mean is:

\[ E(s_1) = E(\Phi_1 s_{t-1}) \]  \hspace{1cm} (F.6)
And the unconditional variance is:

\[
E(s_t s'_t) = E[(\Phi_1 s_{t-1} + \Phi_2 \varepsilon_t)(s'_{t-1} \Phi'_1 + \varepsilon'_2)] \\
= \Phi_1 s_{t-1} s'_{t-1} \Phi'_1 + \Phi_2 \Sigma \Phi'_2
\] (F.7)

Letting \( V \) denote the covariance matrix of \( s \),

\[
V = \Phi_1 V \Phi'_1 + \Phi_2 \Sigma \Phi'_2
\] (F.8)

From which we can recover \( V \),

\[
vec(V) = [I_{\ell^2} - \Phi_1 \otimes \Phi_1]^{-1} \ast vec(\Phi_2 \Sigma \Phi'_2)
\] (F.9)

For a projection of \( d_t \), from the observation equation we have,

\[
\hat{d}_{t|t-1} = \Phi_3 \hat{s}_{t|t-1}
\] (F.10)

And the forecast error is:

\[
d_t - \hat{d}_{t|t-1} = \Phi_3 s_t - \Phi_3 \hat{s}_{t|t-1} = \Phi_3(s_t - \hat{s}_{t|t-1})
\] (F.11)

And associated MSE:

\[
E[(d_t - \hat{d}_{t|t-1})(d_t - \hat{d}_{t|t-1})'] = E[\Phi_3(s_t - s_{t|t-1})(s_t - s_{t|t-1})' \Phi_3]
\] (F.12)

Cross-products disappear and we are left with:

\[
E[(d_t - \hat{d}_{t|t-1})(d_t - \hat{d}_{t|t-1})'] = \Phi_3 V_{t|t-1} \Phi'_3
\] (F.13)

Next, updating the linear projection:

\[
s_{t|t} = s_{t|t-1} + E(s_t - s_{t|t-1})(d_t - \hat{d}_{t|t-1})' [E(d_t - \hat{d}_{t|t-1})(d_t - \hat{d}_{t|t-1})']^{-1} (d_t - \hat{d}_{t|t-1})
\]

\[
= \hat{s}_{t|t-1} + V_{t|t-1} \Phi'_3 (\Phi_3 V_{t|t-1} \Phi'_3)^{-1} (d_t - \hat{d}_{t|t-1})
\]

Then, forecasting:

\[
\hat{s}_{t+1|t} = \Phi_1 \hat{s}_{t|t-1} + \Phi_1 V_{t|t-1} \Phi'_3 (\Phi_3 V_{t|t-1} \Phi'_3)^{-1} (d_t - \hat{d}_{t|t-1})
\] (F.14)

\[
V_{t+1|t} = \Phi_2 \Sigma \Phi'_2 + \Phi_1 V_{t|t-1} \Phi'_1 - \Phi_1 V_{t|t-1} \Phi'_3 (\Phi_3 V_{t|t-1} \Phi'_3)^{-1} \Phi_3 V_{t|t-1} \Phi'_1
\] (F.15)

The log likelihood for \( d_t \), knowing that \( d_t \sim N(\hat{d}_{t|t-1}, \Phi_3 V_{t|t-1} \Phi'_3) \) is written:

\[
lnL = -\frac{NT}{2} \ln(2\pi) - \frac{1}{2} \sum_{t=1}^{T} \ln|\Phi_3 V_t \Phi'_3| - \frac{1}{2} \sum_{t=1}^{T} [(d_t - \hat{d}_{t|t-1})' (\Phi_3 V_t \Phi'_3)(d_t - \hat{d}_{t|t-1})]
\] (F.16)
## Appendix G

### Posterior means and NSE

Table G.1: Posterior means’ point estimate

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Entire sample posterior mean (NSE)</th>
<th>Subsample posterior mean (NSE)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.9877 (0.0012175)</td>
<td>0.98841 (8.9185e-005)</td>
</tr>
<tr>
<td>$\rho_A$</td>
<td>0.99989 (4.4849e-013)</td>
<td>0.99998 (7.3948e-005)</td>
</tr>
<tr>
<td>$\sigma_A$</td>
<td>0.015029 (0.00012892)</td>
<td>0.013418 (0.00035304)</td>
</tr>
<tr>
<td>$\rho_b$</td>
<td>0.99676 (0.00028062)</td>
<td>0.97471 (0.00085411)</td>
</tr>
<tr>
<td>$\sigma_b$</td>
<td>0.013093 (3.536e-005)</td>
<td>0.013063 (4.8457e-005)</td>
</tr>
<tr>
<td>$\rho_z$</td>
<td>0.85604 (0.0058517)</td>
<td>0.7657 (0.0028746)</td>
</tr>
<tr>
<td>$\sigma_z$</td>
<td>0.018068 (0.00022823)</td>
<td>0.014977 (6.9604e-005)</td>
</tr>
<tr>
<td>$\rho_v$</td>
<td>0.28827 (0.013622)</td>
<td>0.37322 (0.0037299)</td>
</tr>
<tr>
<td>$\sigma_v$</td>
<td>0.012468 (0.00010151)</td>
<td>0.014096 (5.2664e-005)</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.36303 (0.0009598)</td>
<td>0.36038 (0.00053023)</td>
</tr>
<tr>
<td>$\phi$</td>
<td>0.033345 (0.00079644)</td>
<td>0.033034 (0.00039341)</td>
</tr>
<tr>
<td>$\eta$</td>
<td>0.70047 (0.00666685)</td>
<td>0.67916 (0.0033531)</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>1.3105 (0.005701)</td>
<td>1.3376 (0.0037022)</td>
</tr>
<tr>
<td>$b$</td>
<td>0.0613 (0.0021902)</td>
<td>0.070124 (0.0017171)</td>
</tr>
<tr>
<td>$\chi$</td>
<td>0.52049 (0.0060713)</td>
<td>0.53195 (0.0042428)</td>
</tr>
<tr>
<td>$\theta$</td>
<td>21.556 (0.29805)</td>
<td>19.616 (0.13101)</td>
</tr>
<tr>
<td>$g$</td>
<td>6.8217 (0.094388)</td>
<td>6.5425 (0.051576)</td>
</tr>
<tr>
<td>$A$</td>
<td>10.048 (0.0010255)</td>
<td>10.047 (6.1826e-005)</td>
</tr>
<tr>
<td>$\rho_\pi$</td>
<td>3131.1 (18.18)</td>
<td>3023.1 (18.714)</td>
</tr>
<tr>
<td>$\rho_g$</td>
<td>0.71299 (0.063402)</td>
<td>1.3926 (0.020655)</td>
</tr>
<tr>
<td>$\rho_\rho$</td>
<td>0.0014543 (0.0002385)</td>
<td>0.0025882 (0.00027827)</td>
</tr>
<tr>
<td>$\pi$</td>
<td>1.2057 (0.019413)</td>
<td>1.5289 (0.017887)</td>
</tr>
<tr>
<td></td>
<td>1.0182 (0.0011923)</td>
<td>1.0107 (0.00081707)</td>
</tr>
</tbody>
</table>
## Appendix H

### Forecasting results

Table H.1: Forecasting results using the entire sample period parameter’s estimates

<table>
<thead>
<tr>
<th>Period</th>
<th>Output</th>
<th>Inflation</th>
<th>Interest rate</th>
<th>Real Money balances</th>
</tr>
</thead>
<tbody>
<tr>
<td>2001Q1</td>
<td>10.752(0.0021)</td>
<td>2.6255(0.3583)</td>
<td>6.1782(0.3093)</td>
<td>10.03 (0.0016)</td>
</tr>
<tr>
<td>2001Q2</td>
<td>10.761(0.0023)</td>
<td>2.2589(0.3406)</td>
<td>5.4271(0.3336)</td>
<td>10.035 (0.0019)</td>
</tr>
<tr>
<td>2001Q3</td>
<td>10.767(0.0025)</td>
<td>0.5612(0.4038)</td>
<td>4.9700(0.4073)</td>
<td>10.047(0.0018)</td>
</tr>
<tr>
<td>2001Q4</td>
<td>10.774(0.0024)</td>
<td>-2.4570(0.5362)</td>
<td>4.3492(0.4011)</td>
<td>10.065(0.0018)</td>
</tr>
<tr>
<td>2002Q1</td>
<td>10.774(0.0035)</td>
<td>-3.1228(0.5833)</td>
<td>2.5032(0.4477)</td>
<td>10.102(0.0032)</td>
</tr>
<tr>
<td>2002Q2</td>
<td>10.768(0.0035)</td>
<td>1.6501(0.3785)</td>
<td>2.8295(0.4246)</td>
<td>10.095(0.0032)</td>
</tr>
<tr>
<td>2002Q3</td>
<td>10.769(0.0034)</td>
<td>4.3905(0.5447)</td>
<td>3.4026(0.3627)</td>
<td>10.077(0.0033)</td>
</tr>
<tr>
<td>2002Q4</td>
<td>10.783(0.0030)</td>
<td>1.3670(0.3941)</td>
<td>3.6078(0.3833)</td>
<td>10.096(0.0027)</td>
</tr>
<tr>
<td>2003Q1</td>
<td>10.782(0.0031)</td>
<td>2.9930(0.4340)</td>
<td>3.4026(0.3725)</td>
<td>10.090(0.0029)</td>
</tr>
<tr>
<td>2003Q2</td>
<td>10.785(0.0033)</td>
<td>3.8957(0.5490)</td>
<td>3.6900(0.3837)</td>
<td>10.081(0.0030)</td>
</tr>
<tr>
<td>2003Q3</td>
<td>10.807(0.0024)</td>
<td>-0.5987(0.4876)</td>
<td>3.8133(0.3453)</td>
<td>10.116(0.0020)</td>
</tr>
<tr>
<td>2003Q4</td>
<td>10.808(0.0030)</td>
<td>2.2182(0.3979)</td>
<td>3.4026(0.3639)</td>
<td>10.113(0.0029)</td>
</tr>
<tr>
<td>2004Q1</td>
<td>10.814(0.0022)</td>
<td>0.9635(0.3294)</td>
<td>3.4846(0.3125)</td>
<td>10.112(0.0019)</td>
</tr>
<tr>
<td>2004Q2</td>
<td>10.821(0.0035)</td>
<td>2.2182(0.4015)</td>
<td>2.8295(0.3441)</td>
<td>10.114(0.0034)</td>
</tr>
<tr>
<td>2004Q3</td>
<td>10.822(0.0038)</td>
<td>2.8295(0.4816)</td>
<td>2.5439(0.4154)</td>
<td>10.12(0.0038)</td>
</tr>
<tr>
<td>2004Q4</td>
<td>10.83(0.0030)</td>
<td>1.8933(0.3658)</td>
<td>2.9930(0.4083)</td>
<td>10.124(0.0029)</td>
</tr>
<tr>
<td>2005Q1</td>
<td>10.839(0.0027)</td>
<td>1.1651(0.3644)</td>
<td>3.2386(0.3802)</td>
<td>10.128(0.0023)</td>
</tr>
</tbody>
</table>

* Note that the interest rate and the inflation rate are expressed in annualized values, whereas output and real-money balances are logged.
Table H.2: Forecasting results using the subsample period parameter’s estimates

<table>
<thead>
<tr>
<th>Period</th>
<th>Output</th>
<th>Inflation</th>
<th>Interest rate</th>
<th>Real Money balances</th>
</tr>
</thead>
<tbody>
<tr>
<td>2001Q1</td>
<td>10.761(0.0024)</td>
<td>1.9745(0.3641)</td>
<td>5.5103(0.1880)</td>
<td>10.029 (0.0017)</td>
</tr>
<tr>
<td>2001Q2</td>
<td>10.763(0.0028)</td>
<td>1.6501(0.4056)</td>
<td>4.8456(0.1958)</td>
<td>10.034 (0.0019)</td>
</tr>
<tr>
<td>2001Q3</td>
<td>10.763(0.0023)</td>
<td>0.0800(0.3647)</td>
<td>4.5145(0.1849)</td>
<td>10.046(0.0017)</td>
</tr>
<tr>
<td>2001Q4</td>
<td>10.762(0.0040)</td>
<td>-2.2763(1.455)</td>
<td>4.0604(0.1948)</td>
<td>10.064(0.0030)</td>
</tr>
<tr>
<td>2002Q1</td>
<td>10.769(0.0027)</td>
<td>-2.7707(0.5369)</td>
<td>2.6255(0.2705)</td>
<td>10.104(0.0021)</td>
</tr>
<tr>
<td>2002Q2</td>
<td>10.777(0.0035)</td>
<td>1.651(0.5038)</td>
<td>2.7479(0.2318)</td>
<td>10.096(0.0024)</td>
</tr>
<tr>
<td>2002Q3</td>
<td>10.781(0.0038)</td>
<td>3.4436(0.6871)</td>
<td>3.1567(0.1922)</td>
<td>10.078(0.0029)</td>
</tr>
<tr>
<td>2002Q4</td>
<td>10.786(0.0027)</td>
<td>0.8829(0.4470)</td>
<td>3.3206(0.1872)</td>
<td>10.096(0.0020)</td>
</tr>
<tr>
<td>2003Q1</td>
<td>10.791(0.0036)</td>
<td>2.2996(0.6813)</td>
<td>3.1976(0.1909)</td>
<td>10.091(0.0029)</td>
</tr>
<tr>
<td>2003Q2</td>
<td>10.799(0.0035)</td>
<td>2.9930(0.6243)</td>
<td>3.3616(0.1647)</td>
<td>10.083(0.0026)</td>
</tr>
<tr>
<td>2003Q3</td>
<td>10.809(0.0020)</td>
<td>-0.8254(0.4111)</td>
<td>3.4846(0.1706)</td>
<td>10.115(0.0015)</td>
</tr>
<tr>
<td>2003Q4</td>
<td>10.813(0.0026)</td>
<td>1.5691(0.4742)</td>
<td>3.1158(0.1682)</td>
<td>10.113(0.0021)</td>
</tr>
<tr>
<td>2004Q1</td>
<td>10.823(0.0024)</td>
<td>0.5210(0.4144)</td>
<td>3.1567(0.1654)</td>
<td>10.112(0.0020)</td>
</tr>
<tr>
<td>2004Q2</td>
<td>10.826(0.0026)</td>
<td>1.5691(0.4745)</td>
<td>2.6255(0.1761)</td>
<td>10.116(0.0020)</td>
</tr>
<tr>
<td>2004Q3</td>
<td>10.831(0.0039)</td>
<td>2.1776(0.7353)</td>
<td>2.3810(0.1903)</td>
<td>10.123(0.0031)</td>
</tr>
<tr>
<td>2004Q4</td>
<td>10.839(0.0031)</td>
<td>1.4074(0.5898)</td>
<td>2.6663(0.1694)</td>
<td>10.124(0.0026)</td>
</tr>
<tr>
<td>2005Q1</td>
<td>10.85(0.0024)</td>
<td>0.7622(0.3865)</td>
<td>2.9521(0.1558)</td>
<td>10.128(0.0018)</td>
</tr>
</tbody>
</table>