Nonlinear preservers

Thèse

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Résumé

Dans cette thèse, nous sommes intéressés par des problèmes de préservation des applications non-linéaires entre deux algèbres de Banach complexes unitaires $\mathscr{A}$ et $\mathscr{B}$. En général, ces problèmes demandent la caractérisation des applications $\varphi : \mathscr{A} \to \mathscr{B}$ non nécessairement linéaires, qui laissent invariant une propriété, une relation ou un sous-ensemble.

Dans le Chapitre 3, la description des applications surjectives φ de $\mathscr{B}(X)$ sur $\mathscr{B}(Y)$, qui satisfont

$$\mathrm{c}(\varphi(S) \pm \varphi(T)) = \mathrm{c}(S \pm T), \qquad (S, T \in \mathscr{B}(X)),$$

est donnée, où $\mathrm{c}(\cdot)$ représente soit le module minimal, ou le module de surjectivité ou le module maximal et $\mathscr{B}(X)$ (resp. $\mathscr{B}(Y)$) dénote l'algèbre de tous les opérateurs linéaires et bornés sur X (resp. sur Y).

Dans le Chapitre 4, une question similaire pour la conorme des opérateurs, est considérée. La caractérisation des applications bicontinues et bijectives φ de $\mathscr{B}(X)$ sur $\mathscr{B}(Y)$, qui satisfont

$$\gamma(\varphi(S \pm \varphi(T)) = \gamma(S \pm T), \qquad (S, T \in \mathscr{B}(X)),$$

est obtenue.

Le Chapitre 5 est consacré à la description des applications surjectives φ_1, φ_2 d'une algèbre de Banach semisimple $\mathscr{A}$ sur une algèbre de Banach $\mathscr{B}$ avec un socle essentiel, qui satisfont

$$\sigma\big(\varphi_1(a)\varphi_2(b)\big) = \sigma(ab), \qquad (a, b \in \mathscr{A}).$$

Aussi, la caractérisation des applications φ de $\mathscr{A}$ sur $\mathscr{B}$, sous les mêmes hypothèses sur $\mathscr{A}$ et $\mathscr{B}$, qui satisfont

$$\sigma\big(\varphi(a)\varphi(b)\varphi(a)\big) = \sigma(aba), \qquad (a, b \in \mathscr{A}),$$

est donnée. Comme conséquences, nous incluons les résultats obtenus au cas des algèbres $\mathscr{B}(X)$ et $\mathscr{B}(Y)$.

Abstract

In this thesis, we are interested in nonlinear preserver problems. In a general formulation, these demand the characterization of a map $\varphi : \mathscr{A} \to \mathscr{B}$, which is not supposed to be linear and leaves a certain property, particular relation, or even a subset invariant, where $\mathscr{A}$ and $\mathscr{B}$ are complex Banach algebras with unit.

In Chapter 3, the description of maps φ from $\mathscr{B}(X)$ onto $\mathscr{B}(Y)$ satisfying

$$\mathrm{c}(\varphi(S) \pm \varphi(T)) = \mathrm{c}(S \pm T), \qquad (S, T \in \mathscr{B}(X)),$$

is given, where $\mathrm{c}(\cdot)$ stands either for the minimum modulus, or the surjectivity modulus, or the maximum modulus and $\mathscr{B}(X)$ (resp. $\mathscr{B}(Y)$) denotes the algebra of all bounded linear operators on a Banach space X (resp. on Y).

In Chapter 4, a similar question for the reduced minimum modulus of operators, is considered. The characterization of bijective bicontinuous maps φ from $\mathscr{B}(X)$ to $\mathscr{B}(Y)$ satisfying

$$\gamma(\varphi(S \pm \varphi(T)) = \gamma(S \pm T), \qquad (S, T \in \mathscr{B}(X)),$$

is obtained.

Chapter 5 is devoted to description of maps φ_1, φ_2 from a semisimple Banach algebra $\mathscr{A}$ onto a Banach algebra $\mathscr{B}$ with an essential socle, that satisfy

$$\sigma\big(\varphi_1(a)\varphi_2(b)\big) = \sigma(ab), \qquad (a, b \in \mathscr{A}).$$

Also, the characterization of maps φ from $\mathscr{A}$ onto $\mathscr{B}$, under the same assumptions on $\mathscr{A}$ and $\mathscr{B}$, satisfying

$$\sigma\big(\varphi(a)\varphi(b)\varphi(a)\big) = \sigma(aba), \qquad (a, b \in \mathscr{A}),$$

is given. The corollaries for algebras $\mathscr{B}(X)$ and $\mathscr{B}(Y)$, that follow immediately from the results, are included.

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Avant-propos

Cette thèse étudie des problèmes de préservation des applications non-linéaires. Les résultats sont présentés dans les trois articles, qui sont inclus dans la thèse comme des chapitres. Le Chapitre 3 est constitué d'un article coécrit avec mon directeur de thèse à l'Université Laval, Javad Mashreghi, et mon codirecteur de thèse de l'Université de Syracuse, Abdellatif Bourhim. Cet article est intitulé **Nonlinear maps preserving the minimum and surjectivity moduli** et publié dans *Linear Algebra and its Applications* en décembre 2014. Le Chapitre 4 est basé sur un article coécrit avec Javad Mashreghi, intitulé **Nonlinear maps preserving the reduced minimum modulus of operators** et publié dans *Linear Algebra and its Applications* en mars 2016. Le Chapitre 5 repose sur un article s'intitulant **Maps between Banach algebras preserving the spectrum**, coécrit avec Abdellatif Bourhim et Javad Mashreghi et resoumis pour publication, après la réalisation des modifications suggérées par deux arbitres, dans *Archiv der Mathematik* en juin 2016.

Chapter 1

Introduction

1.1 Basic definitions and properties

In this section we introduce definitions and properties of concepts that are used in the thesis.

We denote complex Banach algebras with unit by $\mathscr{A}$ and $\mathscr{B}$, $\mathrm{Inv}(\mathscr{A})$ is the set of all invertible elements of $\mathscr{A}$ and

$$\sigma(a) := \{\lambda \in \mathbb{C} : a - \lambda \mathbf{1} \notin \mathrm{Inv}(\mathscr{A})\}$$

is the *spectrum* of $a \in \mathscr{A}$, which is a nonempty compact set and

$$\mathrm{r}(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}$$

is the *spectral radius* of $a \in \mathscr{A}$. An element $a \in \mathscr{A}$ is called idempotent if $a^2 = a$. Also, an element $a \in \mathscr{A}$ is said to be *nilpotent* if there is $n \in \mathbb{N}$ such that $a^n = 0$, and it is called *quasi-nilpotent* if $\mathrm{r}(a) = 0$.

We denote by $\mathrm{rad}\mathscr{A}$ the *Jacobson radical* of $\mathscr{A}$, which is the intersection of all maximal left (or right) ideals of $\mathscr{A}$. The Banach algebra $\mathscr{A}$ is called *semisimple* if its Jacobson radical is zero, which is equivalent to the fact that $a = 0$ is the only element in $\mathscr{A}$, for which $\mathrm{r}(xa) = 0$ for all $x \in \mathscr{A}$. We need the spectral characterization of the radical, which was given by Zemánek [6, Theorem 5.3.1].

Theorem 1.1.1. *Let $\mathscr{A}$ and $\mathscr{B}$ be Banach algebras. Then the following properties are equivalent.*

(i) *a is in the Jacobson radical of $\mathscr{A}$.*

(ii) $\sigma(a+x) = \sigma(x)$, *for all $x \in \mathscr{A}$.*

(iii) $\mathrm{r}(a+x) = 0$, *for all quasi-nilpotent elements $x \in \mathscr{A}$.*

We say that $\mathscr{A}$ is a *prime* if for any two elements $x, y \in \mathscr{A}$, the identity $x\mathscr{A}y = \{0\}$ holds true when either $x = 0$ or $y = 0$.

A *Jordan algebra* is an algebra with a bilinear multiplication that satisfies the two conditions $xy = yx$ and $(xy)x^2 = x(yx^2)$, for all x, y in the algebra. Given an associative algebra we can build a

Jordan algebra, defining the *Jordan product* $x \circ y = \frac{1}{2}(xy + yx)$. A *Jordan-Banach algebra* is a Jordan algebra with a complete norm satisfying $\|x \circ y\| \leq \|x\| \|y\|$. Jordan-Banach algebras are considered over the complex field and have a unit.

The *socle* of $\mathscr{A}$, which is denoted by $\mathrm{Soc}(\mathscr{A})$, is defined as the sum of all minimal left (or right) ideals of $\mathscr{A}$, if $\mathscr{A}$ has a minimal left (or right) ideal, and as $\mathrm{Soc}(\mathscr{A}) = \{0\}$ otherwise. We say that a nonzero element $u \in \mathscr{A}$ has rank one if u belongs to some minimal left (or right) ideal of $\mathscr{A}$. This is equivalent to the fact that for all $x \in \mathscr{A}$, the spectrum $\sigma(ux)$ contains at most one nonzero scalar. Note that an element of the socle can be written as a sum of elements of rank one and for $a \in \mathrm{Soc}(\mathscr{A})$ the spectrum $\sigma(ax)$ is finite for all $x \in \mathscr{A}$. We refer the reader to [6, 9, 26] for proofs and more details.

An ideal I of $\mathscr{A}$ is said to be *essential* if it has a nonzero intersection with every nonzero ideal of $\mathscr{A}$. If $\mathscr{A}$ is semisimple, then I is essential if and only if 0 is the only element a of $\mathscr{A}$ for which $aI = 0$.

Let $\mathscr{A}$ be a unital complex Banach algebra. A map $^* : \mathscr{A} \to \mathscr{A}$ is called an *involution* if it satisfies

(i) $(x^*)^* = x$, for every $x \in \mathscr{A}$,

(ii) $(xy)^* = y^* x^*$ and $(x + y)^* = x^* + y^*$ for every $x, y \in \mathscr{A}$,

(iii) $(\lambda x)^* = \overline{\lambda} x^*$, for every $\lambda \in \mathbb{C}$ and $x \in \mathscr{A}$,

(iv) $\mathbf{1}^* = \mathbf{1}$

If also $\|xx^*\| = \|x\|^2$, for every $x \in \mathscr{A}$, then we say that $\mathscr{A}$ is a *C^*-algebra*.

Let $\mathscr{H}$ be an infinite-dimensional complex Hilbert space and $\mathscr{B}(\mathscr{H})$ be the algebra of all bounded linear operators on $\mathscr{H}$. $\mathscr{B}(\mathscr{H})$ is the prototypical example of a C^*-algebra, where as an involution of an operator of $\mathscr{B}(\mathscr{H})$ is taken its adjoint operator. By Gelfand–Neumark theorem [36] we know that, conversely, every C^*-algebra is isomorphic to a norm-closed adjoint-closed subalgebra of $\mathscr{B}(\mathscr{H})$ for a suitable Hilbert space $\mathscr{H}$. We remark also that a C^*-algebra is semisimple, as $0 = \mathrm{r}(xx^*) = \|x\|^2$ and $x = 0$.

An element h of a Banach algebra $\mathscr{A}$ is called *hermitian* if it has real numerical range, that is, if $f(h) \in \mathbb{R}$, for all f in the Banach dual space of $\mathscr{A}$ with $f(\mathbf{1}) = 1 = \|f\|$. We denote by $H(\mathscr{A})$ the closed real subspace of $\mathscr{A}$ of all hermitian elements of $\mathscr{A}$. It is well known that an element in a C^*-algebra is hermitian if and only if it is self-adjoint. By the Vidav-Palmer theorem, a unital Banach algebra $\mathscr{A}$ is a C^*-algebra if and only if it is generated by its hermitian elements, that is if $\mathscr{A} = H(\mathscr{A}) + iH(\mathscr{A})$ and the involution is given by $(h + ik)^* := h - ik$, where $h, k \in H(\mathscr{A})$; see [12].

A weakly closed, unital *-subalgebra of $\mathscr{B}(\mathscr{H})$ is called a *von Neumann algebra*. Clearly $\mathscr{B}(\mathscr{H})$ itself is a von Neumann algebra and every von Neumann algebra is a C^* algebra. There is also an equivalent abstract definition of a von Neumann algebra, given by Sakai [73]. It can be defined as a C^*-algebra, which as a Banach space is the dual of some other Banach space called the *predual*. Therefore, a *Jordan-von Neumann algebra* is a Jordan algebra, which has a predual.

Let X and Y be infinite-dimensional complex Banach spaces and $\mathscr{B}(X,Y)$ be the space of all bounded linear maps from X to Y. As usual, we write $\mathscr{B}(X)$ instead of $\mathscr{B}(X,X)$. We remark that $\mathscr{B}(X)$ is a semisimple prime algebra and $\mathrm{Soc}(\mathscr{B}(X))$ is equal to the ideal of all finite rank operators in $\mathscr{B}(X)$, which we denote by $\mathscr{F}(X)$. The dual space of X is denoted by X^* and the Banach space adjoint of an operator $T \in \mathscr{B}(X)$ is denoted by T^*.

Now, we give the definitions of different types of moduli of an operator $T \in \mathscr{B}(X)$. The *minimum modulus* T is

$$\mathrm{m}(T) := \inf\{\|Tx\| : x \in X,\ \|x\| = 1\},$$

and is positive if and only if T is bounded below, which means T is injective and has a closed range. The *surjectivity modulus* of T is

$$\mathrm{q}(T) := \sup\{\varepsilon \geq 0 : \varepsilon B_X \subseteq T(B_X)\},$$

where B_X is the closed unit ball of X. It is positive precisely when T is surjective. And the *maximum modulus* of T is

$$\mathrm{M}(T) := \max\{\mathrm{m}(T), \mathrm{q}(T)\}$$

and obviously, is positive whether T is bounded below or T is surjective. And also, we define the *reduced minimum modulus* of T by the following

$$\gamma(T) := \begin{cases} \inf\{\|Tx\| : \operatorname{dist}(x, Ker\,T) \geq 1\} & \text{if } T \neq 0, \\ \infty & \text{if } T = 0, \end{cases}$$

which is positive precisely when T has closed range. We have the following equalities for minimum and surjectivity moduli

$$\mathrm{m}(T) = \inf\{\|TS\| : S \in \mathscr{B}(X),\ \|S\| = 1\} \quad \text{and} \quad \mathrm{q}(T) = \inf\{\|ST\| : S \in \mathscr{B}(X),\ \|S\| = 1\}. \tag{1.1.1}$$

It is easy to see that

$$\mathrm{m}(T^*) = \mathrm{q}(T) \quad \text{and} \quad \mathrm{q}(T^*) = \mathrm{m}(T).$$

Another important property is that for every invertible operator $T \in \mathscr{B}(X)$ we have

$$\mathrm{m}(T) = \mathrm{q}(T) = \mathrm{M}(T) = \gamma(T) = \|T^{-1}\|^{-1}. \tag{1.1.2}$$

We remark that $\mathrm{M}(T) \leq \gamma(T)$ and $\mathrm{M}(T) = \gamma(T)$, if $\mathrm{M}(T) > 0$.

The equalities (1.1.1) opened the way to generalize the definitions of moduli to the general Banach algebra case. Let a be an element in a Banach algebra $\mathscr{A}$. Then

$$\mathrm{m}(a) := \mathrm{m}(L_a) = \inf\{\|ax\| : x \in \mathscr{A},\ \|x\| = 1\}, \quad \mathrm{q}(a) := \mathrm{m}(R_a) = \inf\{\|xa\| : x \in \mathscr{A},\ \|x\| = 1\},$$

$$\mathrm{M}(a) := \max\{\mathrm{m}(a), \mathrm{q}(a)\} \quad \text{and} \quad \gamma(a) := \begin{cases} \inf\{\|ax\| : \operatorname{dist}(x, Ker\,L_a) \geq 1\} & \text{if } a \neq 0, \\ \infty & \text{if } a = 0, \end{cases}$$

where L_a and R_a are the left and right multiplication operators by a. We have the same property as (1.1.2) for an invertible element $a \in \operatorname{Inv}\mathscr{A}$ in a general case too.

$$\mathrm{m}(a) = \mathrm{q}(a) = \mathrm{M}(a) = \gamma(a) = \|a^{-1}\|^{-1}.$$

And it is easy to prove that for every $a, b \in \mathscr{A}$

$$\mathrm{m}(a)\mathrm{m}(b) \le \mathrm{m}(ab) \le \|a\|\mathrm{m}(b) \qquad \text{and} \qquad \mathrm{q}(a)\mathrm{q}(b) \le \mathrm{q}(ab) \le \mathrm{q}(a)\|b\|.$$

Also we have for every $a, b \in \mathscr{A}$

$$|\mathrm{c}(a) - \mathrm{c}(b)| \le \|a - b\|,$$

where $\mathrm{c}(\cdot)$ stands either for the minimum modulus, or the surjectivity modulus, or the maximum modulus. And so, we conclude that the spectral functions $\mathrm{m}(\cdot)$, $\mathrm{q}(\cdot)$ and $\mathrm{M}(\cdot)$ are contractive, hence continuous. However, we must note that $\gamma(\cdot)$ is not continuous as the following example shows:

$$\gamma\left(\begin{bmatrix} 1 & 0 \\ 0 & 1/n \end{bmatrix}\right) = 1/n \to 0 \ne 1 = \gamma\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) \qquad \text{as } \ n \to \infty.$$

Particularly, for the case of the Banach algebra $\mathscr{B}(X)$ we obtain that the set $\{T \in \mathscr{B}(X) : \mathrm{c}(T) > 0\}$ is open, so the subset of all lower bounded, surjective and lower bounded or surjective operators in $\mathscr{B}(X)$ is open, which are denoted respectfully by $\Omega_{\text{LB}}(X)$, $\Omega_{\text{Surj}}(X)$ and $\Omega_{\text{LB-or-Surj}}(X)$.

If particularly $\mathscr{A}$ is a C^*-algebra and $a \in \mathscr{A}$, then by [15, Lemma 2.2]

$$\mathrm{m}(a) = \inf\{\lambda : \lambda \in \sigma(|a|)\} \qquad \text{and} \qquad \mathrm{q}(a) = \inf\{\lambda : \lambda \in \sigma(|a^*|)\}, \tag{1.1.3}$$

where $|a| := (a^* a)^{1/2}$ is the absolute value of a. It follows that

$$\mathrm{m}(a) = \mathrm{q}(a^*)$$

and consequently $\mathrm{M}(a) = \mathrm{M}(a^*)$. If the algebra $\mathscr{A}$ has a property that every left (right) invertible element is invertible, then in fact we have the equality of the minimum modulus and the surjectivity modulus of an element. This is the case for the algebra $M_n(\mathbb{C})$ of all $n \times n$ complex matrices.

The formula for the reduced minimum modulus is obtained by Harte and Mbekhta in [39], which is

$$\gamma(a)^2 = \gamma(a^* a) = \inf\left\{\lambda : \lambda \in \sigma(a^* a) \setminus \{0\}\right\} = \gamma(a^*)^2. \tag{1.1.4}$$

Now, let us give some definitions about a mapping $\varphi : \mathscr{A} \to \mathscr{B}$. We say that

- φ is *unital* if

$$\varphi(\mathbf{1}_{\mathscr{A}}) = \mathbf{1}_{\mathscr{B}},$$

- φ *preserves the spectrum* if for every $x \in \mathscr{A}$

$$\sigma(\varphi(x)) = \sigma(x),$$

- φ *compresses the spectrum* if for every $x \in \mathscr{A}$

$$\sigma(\varphi(x)) \subset \sigma(x),$$

- φ *expands the spectrum* if for every $x \in \mathscr{A}$

$$\sigma(\varphi(x)) \supset \sigma(x),$$

- φ *preserves invertibility* if for every $x \in \mathscr{A}$

$$x \text{ is invertible } \Longrightarrow \varphi(x) \text{ is invertible,}$$

- φ *preserves invertibility in both directions* if for every $x \in \mathscr{A}$

$$x \text{ is invertible } \Longleftrightarrow \varphi(x) \text{ is invertible,}$$

- φ is a *spectral isometry* if for every $x \in \mathscr{A}$

$$\mathrm{r}(\varphi(x)) = \mathrm{r}(x),$$

- φ is *spectrally bounded* if there is a constant M such that for every $x \in \mathscr{A}$

$$\mathrm{r}(\varphi(x)) \leq M\mathrm{r}(x),$$

- φ is *spectrally bounded from below* if there is a constant m such that for every $x \in \mathscr{A}$

$$\mathrm{r}(\varphi(x)) \geq m\mathrm{r}(x).$$

We mention here an interesting fact, which is cited from [6, Theorem 5.5.2].

Theorem 1.1.2. *Let $\mathscr{A}$ be a Banach algebra and let $\mathscr{B}$ be a semisimple Banach algebra. Suppose that $\varphi : \mathscr{A} \to \mathscr{B}$ is a surjective linear map such that $r(\varphi(x)) \leq r(x)$ for all $x \in \mathscr{A}$. Then φ is continuous.*

Clearly, if φ preserves (resp. compresses, expands) the spectrum, then φ is a spectral isometry (resp. spectrally bounded, spectrally bounded from below).

Also, it is easy to see that if φ is a unital linear map that preserves the invertibility (in both directions) then φ compresses (preserves) the spectrum. Conversely, a linear map that compresses (preserves) the spectrum, obviously preserves the invertibility (in both directions) and by the following lemma, cited from [5, Lemme 4, p.30], we can assume that it is unital.

Lemma 1.1.3. *Let $\mathscr{A}$ and $\mathscr{B}$ be two Banach algebras with unit. If $\varphi : \mathscr{A} \to \mathscr{B}$ is a linear map that compresses (preserves) the spectrum, then $\phi(x) := (\varphi(\mathbf{1}))^{-1}\varphi(x)$ is a linear map that also compresses (preserves) the spectrum and is unital.*

But we can claim more when $\mathscr{A}$ is semisimple and $\varphi : \mathscr{A} \to \mathscr{B}$ is a linear spectrum-preserving map. We also write down the proof, cited from [6, 52], as it is interesting and short.

Lemma 1.1.4. *Let $\mathscr{A}$ be a semisimple Banach algebra. Suppose that $\varphi : \mathscr{A} \to \mathscr{B}$ is a linear spectrum-preserving map. Then we have that*

(i) φ is injective;

(ii) if φ is surjective then φ is unital;

(iii) if φ is surjective then φ is continuous, consequently there exist two positive constants α, β such that $\alpha\|x\| \le \|\varphi(x)\| \le \beta\|x\|$, for all $x \in \mathscr{A}$.

Proof. (i) Suppose that $\varphi(a) = 0$. Then $\sigma(a+x) = \sigma(\varphi(a+x)) = \sigma(\varphi(x)) = \sigma(x)$ for every $x \in \mathscr{A}$. By Zemánek's characterization of the radical, we have $a \in \mathrm{rad}\mathscr{A} = \{0\}$.

(ii) Since φ is surjective, there exists an $a \in \mathscr{A}$ such that $\varphi(a) = \mathbf{1}$. We have that $\sigma(x - \mathbf{1} + a) = \sigma(\varphi(x - \mathbf{1}) + \mathbf{1}) = \sigma(x - \mathbf{1}) + 1 = \sigma(x)$ for every $x \in \mathscr{A}$. Again, by Zemánek's characterization of the radical $a - \mathbf{1} \in \mathrm{rad}\mathscr{A} = \{0\}$, so $a = \mathbf{1}$.

(iii) In particular we have $\mathrm{r}(\varphi(x)) = \mathrm{r}(x)$, for all $x \in \mathscr{A}$ and moreover φ is surjective, so, we can use the Theorem 1.1.2 and conclude that φ is continuous. By (i), φ is also injective, so by the open mapping theorem, φ^{-1} is continuous.

□

We say that a linear map $\varphi : \mathscr{A} \to \mathscr{B}$ is a *homomorphism* if

$$\varphi(ab) = \varphi(a)\varphi(b), \qquad (a, b \in \mathscr{A}).$$

It is called an *anti-homomorphism* if

$$\varphi(ab) = \varphi(b)\varphi(a), \qquad (a, b \in \mathscr{A}).$$

And it is said to be a *Jordan homomorphism* if

$$\varphi(ab + ba) = \varphi(a)\varphi(b) + \varphi(b)\varphi(a), \qquad (a, b \in \mathscr{A}),$$

which is equivalent to the following condition

$$\varphi(a^2) = (\varphi(a))^2, \qquad (a \in \mathscr{A}).$$

If $\mathscr{A}$ and $\mathscr{B}$ are C^*-algebras, then a linear map $\varphi : \mathscr{A} \to \mathscr{B}$ is called a Jordan *-homomorphism if it is a Jordan homomorphism and

$$\varphi(a^*) = (\varphi(a))^* \qquad (a \in \mathscr{A}).$$

Evidently, every homomorphism or anti-homomorphism is a Jordan homomorphism. The converse is true in some case. Particularly, a Jordan homomorphism $\varphi : \mathscr{A} \to \mathscr{B}$ is a homomorphism or an anti-homomorphism if $\mathscr{B}$ is an integral domain, by [51], or when $\mathscr{B}$ is a prime algebra, by [44]. For example, as we know, a Banach algebra $\mathscr{B}(X)$ of all bounded linear operators on a Banach space X is prime, so a Jordan homomorphism $\varphi : \mathscr{B}(X) \to \mathscr{B}(Y)$ is either a homomorphism or an anti-homomorphism.

1.2 Kaplansky's conjecture and some partial results

In *preserver problems* the characterization of a map $\varphi : \mathscr{A} \to \mathscr{B}$ between Banach algebras $\mathscr{A}$ and $\mathscr{B}$ is demanded, considering that it leaves a certain property, a particular relation, or even a subset invariant. We call it *linear preserver problems* when φ is supposed to be linear and we call it *nonlinear preserver problems* when we don't assume any algebraic condition like linearity or additivity or multiplicativity for φ. In all cases that have been studied by now, nonlinear preservers are proved to be linear or conjugate linear perhaps plus an element of the target algebra.

In this section we will provide some historical facts and a few important results of preserver problems.

The first results are about the preservers of matrices. As usual, we denote by $M_n(\mathbb{C})$ the algebra of all $n \times n$ complex matrices. In 1897, Frobenius [34] characterized linear maps $\varphi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ that preserve the determinant of matrices. Later, in 1959, Marcus and Moyls [62] improved this result. They described the cases when a linear map $\varphi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ preserves the rank of all matrices, or the determinant or the spectrum of all hermitian matrices. In the last case the representation of φ is the following

$$\varphi(T) = UTU^{-1} \quad \text{or} \quad \varphi(T) = UT^{tr}U^{-1}, \qquad (T \in M_n(\mathbb{C})),$$

where U is a non-singular matrix and T^{tr} is the transpose of T. We remark here that the extensions of this result to the case of arbitrary field $\mathbb{F}$ instead of the complex numbers $\mathbb{C}$ have been obtained recently; see [2, 32, 75]

Almost simultaneously, in 1967-1968, Gleason, Kahane and Żelazko [37, 55] have demonstrated that if $\mathscr{A}$ is a commutative Banach algebra and $\mathscr{B}$ is a semisimple commutative Banach algebra, then every unital, invertibility preserving linear map $\varphi : \mathscr{A} \to \mathscr{B}$ is a homomorphism. After that, in the same year, Żelazko [83] have showed that the assumption of commutativity of $\mathscr{A}$ can be removed.

Theorem 1.2.1. *Let $\mathscr{A}$ be a Banach algebra and $\mathscr{B}$ be a semisimple commutative Banach algebra. Then every unital, invertibility preserving linear map $\varphi : \mathscr{A} \to \mathscr{B}$ is a homomorphism.*

In 1970, motivated by these results, Kaplansky [56] asked *when must an invertibility preserving linear map $\varphi : \mathscr{A} \to \mathscr{B}$ be a Jordan homomorphism.* In this general form, the question does not have a positive answer. Example 1.2.2 shows that without the surjectivity, φ need not to be a Jordan homomorphism. Example 1.2.3 shows that without the semisimplicity of $\mathscr{A}$ and $\mathscr{B}$, the map φ need not be a Jordan homomorphism.

Example 1.2.2. *Let $\varphi : \mathscr{M}_2(\mathbb{C}) \to \mathscr{M}_4(\mathbb{C})$ be defined by*

$$\varphi(X) = \begin{bmatrix} X & X - X^{tr} \\ 0 & X \end{bmatrix}, \qquad (X \in \mathscr{M}_2(\mathbb{C})).$$

Then φ is a unital linear map such that $\sigma(\varphi(X)) = \sigma(X)$, i.e. φ is invertibility preserving. But

$$\varphi(X^2) - (\varphi(X))^2 = \begin{bmatrix} 0 & (X - X^{tr})^2 \\ 0 & 0 \end{bmatrix}, \qquad (X \in \mathscr{M}_2(\mathbb{C})).$$

Hence, φ is not a Jordan homomorphism.

Example 1.2.3. *Consider the Banach algebra*

$$\mathscr{A} = \left\{ \begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix} : X, Y, Z \in \mathscr{M}_2(\mathbb{C}) \right\}.$$

Since

$$\sigma\left(\begin{bmatrix} 0 & Z \\ 0 & 0 \end{bmatrix} \right) = \{0\}, \qquad (Z \in \mathscr{M}_2(\mathbb{C})),$$

the algebra $\mathscr{A}$ is not semisimple. Now, define the linear map $\varphi : \mathscr{A} \to \mathscr{A}$ by

$$\varphi(W) := \begin{bmatrix} X & Z \\ 0 & Y^{tr} \end{bmatrix}, \quad \textit{for} \quad W = \begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix} \in \mathscr{A}.$$

We have $\sigma(\varphi(W)) = \sigma(W)$ and φ is unital, i.e. φ is invertibility preserving, but

$$\varphi(W^2) - (\varphi(W))^2 = \begin{bmatrix} 0 & Z(Y - Y^{tr}) \\ 0 & 0 \end{bmatrix}, \qquad (W \in \mathscr{A}).$$

Hence, φ is not a Jordan homomorphism.

In the light of the preceding two examples and several partial positive results, Kaplansky's question was reformulated by Aupetit [7] as follows.

Conjecture 1.2.4 (Kaplansky's Conjecture)**.** *Let $\mathscr{A}$ and $\mathscr{B}$ be semisimple Banach algebras. Then every surjective unital linear map $\varphi : \mathscr{A} \to \mathscr{B}$ that preserves invertibility is a Jordan homomorphism.*

Let us remark that the converse of this conjecture is true even for a more general case. If $\mathscr{A}$ and $\mathscr{B}$ are complex Banach algebras with unit and $\varphi : \mathscr{A} \to \mathscr{B}$ is a surjective Jordan homomorphism, then φ is unital and preserves invertibility; see for instance [79, Proposition 1.3].

This conjecture can be observed as a spectrum compressing problem. There is also another variant of Kaplansky's conjecture, where a spectrum preserving problem is considered.

Conjecture 1.2.5. *Let $\mathscr{A}$ and $\mathscr{B}$ be semisimple Banach algebras. Then every surjective linear map $\varphi : \mathscr{A} \to \mathscr{B}$ that preserves the spectrum is a Jordan isomorphism.*

Kaplansky's conjecture is still open even when $\mathscr{A}$ and $\mathscr{B}$ are C^*-algebras, but for some special cases the answer is positive and there are many partial results. Let us introduce the most important ones.

A useful way to treat Kaplansky's question for Banach algebras of all bounded linear operators on a Banach space is to reduce it to the problem of characterizing linear maps preserving rank one operators. That is what Jafarian and Sourour did in [52] and answered positively Kaplansky's question for this case; see Theorem 1.2.8. Let us mention the steps that they followed. First they proved the following lemma, which is a spectral characterization of rank one operators.

Lemma 1.2.6 (Jafarian-Sourour [52], 1986)**.** *Let $R \in \mathscr{B}(X)$, $R \neq 0$. The following conditions are equivalent.*

(i) R has rank 1.

(ii) $\sigma(T+R) \cap \sigma(T+\lambda R) \subseteq \sigma(T)$ for every $T \in \mathscr{B}(X)$ and every $\lambda \in \mathbb{C}$ such that $\lambda \neq 1$.

If we have a surjective spectrum preserving linear map $\varphi : \mathscr{B}(X) \to \mathscr{B}(Y)$, then with the aid of Lemma 1.2.6 we conclude that φ maps rank one operators to rank one operators. We remind that a rank one operator in $\mathscr{B}(X)$ has the following form

$$(x \otimes f)(y) := \langle y, f \rangle x, \qquad (y \in X),$$

for some nonzero $x \in X$ and $f \in X^*$, where $\langle y, f \rangle$ stands for $f(y)$. The next step in [52] was characterizing maps preserving operators of rank one. We present it here, because it is an important result itself.

Theorem 1.2.7 (Jafarian-Sourour [52], 1986)**.** *If a bijective linear map $\varphi : \mathscr{F}(X) \to \mathscr{F}(Y)$ preserves rank one operators, then one of following situations hold.*

(i) There exist isomorphisms $A \in \mathscr{B}(X, Y)$ and $B \in \mathscr{B}(X^*, Y^*)$ such that

$$\varphi(x \otimes f) = Ax \otimes Bf, \qquad x \in X, f \in X^*.$$

(ii) There exist isomorphisms $C \in \mathscr{B}(X^*, Y)$ and $D \in \mathscr{B}(X, Y^*)$ such that

$$\varphi(x \otimes f) = Cf \otimes Dx, \qquad x \in X, f \in X^*.$$

And finally, with the aid of this characterization, Jafarian and Sourour proved the following theorem, which is a positive answer to the Kaplansky's conjecture for the case of Banach algebras of bounded linear operators on a Banach space.

Theorem 1.2.8 (Jafarian-Sourour [52], 1986)**.** *A surjective linear map $\varphi : \mathscr{B}(X) \to \mathscr{B}(Y)$ preserves the spectrum if and only if φ has one of the following forms.*

(i) There exists an isomorphism $A \in \mathscr{B}(X, Y)$ such that

$$\varphi(T) = ATA^{-1}, \qquad (T \in \mathscr{B}(X)).$$

(ii) There exists an isomorphism $B \in \mathscr{B}(X^*, Y)$ such that

$$\varphi(T) = BT^*B^{-1}, \qquad (T \in \mathscr{B}(X)).$$

This case can occur only if both X and Y are reflexive.

Let us explain the last sentence. If φ takes form (ii), the surjectivity of φ implies that every operator on X^* is a dual of an operator on X. So, if we consider rank one operators $f \otimes G : X^* \to X^*$ with fixed nonzero $f \in X^*$ and arbitrary $G \in X^{**}$, we can say that there exists $T \in \mathscr{B}(X)$ such that $G(\varphi)f(x) = (T^*(\varphi))(x) = \varphi(Tx)$ for all $x \in X$ and $\varphi \in X^*$. If we fix x_0 such that $f(x_0) = 1$, we obtain that $G(\varphi) = \varphi(Tx_0)$ for all $\varphi \in X^*$, which means that X is reflexive. As Y must be isomorphic to X^*, so Y is reflexive too.

This result has been extended in many ways, but we would like to mention just a few of them. In [23] Brešar and Šemrl characterized surjective linear spectral isometries $\varphi : \mathscr{B}(X) \to \mathscr{B}(X)$.

Theorem 1.2.9 (Brešar-Šemrl [23], 1996)**.** *A surjective linear map $\varphi : \mathscr{B}(X) \to \mathscr{B}(X)$ is a spectral isometry if and only if there is $\alpha \in \mathbb{C}$ of modulus one such that φ has one of the following forms.*

(i) There exists an isomorphism $A \in \mathscr{B}(X)$ such that

$$\varphi(T) = \alpha ATA^{-1}, \qquad (T \in \mathscr{B}(X)).$$

(ii) There exists an isomorphism $B \in \mathscr{B}(X^*, X)$ such that

$$\varphi(T) = \alpha BT^*B^{-1}, \qquad (T \in \mathscr{B}(X)).$$

This case can occur only if X is reflexive.

To prove this theorem they showed that φ preserves nilpotents in both directions and after they applied the result of Šemrl [74], where the description of all surjective linear maps preserving nilpotent operators in both directions on a Banach space X is given. In [33] Fošner and Šemrl improved this result for the case of surjective linear operators that are both spectrally bounded and spectrally bounded from bellow.

Theorem 1.2.10 (Fošner-Šemrl [33], 2005)**.** *Let $\varphi : \mathscr{B}(X) \to \mathscr{B}(X)$ be a surjective linear map. Suppose that there exist positive constants m and M such that $mr(T) \le r(\varphi(T)) \le Mr(T)$ for every $T \in \mathscr{B}(X)$. Then there exist a spectrally bounded linear functional $f : \mathscr{B}(X) \to \mathbb{C}$ and a nonzero complex number $\alpha \in \mathbb{C}$ such that φ has one of the following forms.*

(i) There exists an isomorphism $A \in \mathscr{B}(X, Y)$ such that

$$\varphi(T) = \alpha ATA^{-1} + f(T)I, \qquad (T \in \mathscr{B}(X)).$$

(ii) There exists an isomorphism $B \in \mathscr{B}(X^*, Y)$ such that

$$\varphi(T) = \alpha B T^* B^{-1} + f(T) I, \qquad (T \in \mathscr{B}(X)).$$

This case can occur only if X is reflexive.

Let us make some comments about this result. First, when $X = \mathscr{H}$ is an infinite dimensional Hilbert space, the second term disappears, as by [64, Corollary 3.9] the only spectrally bounded linear functional $f : \mathscr{B}(\mathscr{H}) \to \mathbb{C}$ is the zero functional. But this is not true for general Banach spaces. We know that there exist an infinite dimensional Banach space X such that $\mathscr{B}(X)$ has a nonzero multiplicative linear functional. It maps every element of the algebra into its spectrum, so it is spectrally bounded. The converse of this theorem in general is not true. If φ takes one of the forms, then obviously φ is spectrally bounded, but it does not need to be surjective nor spectrally bounded below. However, it turns out that if $\alpha \neq -f(I)$, then φ is bijective and spectrally bounded below. The details are in the article [33].

Another extension of the Theorem 1.2.8 was obtained by Sourour in [79]. He characterized surjective linear maps $\varphi : \mathscr{B}(X) \to \mathscr{B}(Y)$ that preserve invertibility in just one direction.

Theorem 1.2.11 (Sourour [79], 1996)**.** *Let $\varphi : \mathscr{B}(X) \to \mathscr{B}(Y)$ be a unital surjective linear map that preserves invertibility. Then either φ vanishes on $\mathscr{F}(X)$ or φ is injective. In the latter case φ has one of the following forms.*

(i) There exist isomorphisms $A \in \mathscr{B}(X, Y)$ and $B \in \mathscr{B}(Y, X)$ such that

$$\varphi(T) = ATB, \qquad (T \in \mathscr{B}(X)).$$

(ii) There exist isomorphisms $A \in \mathscr{B}(X^*, Y)$ and $B \in \mathscr{B}(Y, X^*)$ such that

$$\varphi(T) = AT^* B, \qquad (T \in \mathscr{B}(X)).$$

This case can occur only if both X and Y are reflexive.

First we note that if φ is unital then $B = A^{-1}$. Then, we remark that although, by Lemma 1.1.4, the linear spectrum-preserving maps $\varphi : \mathscr{B}(X) \to \mathscr{B}(Y)$ are necessarily injective, it is not true for invertibility-preserving maps. Recently, Argyros and Haydon [4] have constructed a Banach space X such that $\mathscr{B}(X) = \mathbb{C}\mathbf{1} + \mathscr{K}(X)$, where $\mathscr{K}(X)$ denotes the closed ideal of all compact operators on X. For this Banach space X, the following linear functional on $\mathscr{B}(X)$ preserves invertibility, vanishes on $\mathscr{K}(X)$ and is not injective:

$$f(\lambda \mathbf{1} + K) = \lambda, \qquad (K \in \mathscr{K}(X)).$$

However, as it was shown in [79], if $X = \mathscr{H}$ is a separable infinite-dimensional Hilbert space, then a surjective linear spectrum-preserving map $\varphi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(Y)$ is injective.

There is an improvement of Lemma 1.2.6 on spectral characterization of rank one operators in [79].

Theorem 1.2.12 (Sourour [79], 1996)**.** *For an operator $R \in \mathscr{B}(X)$, the following conditions are equivalent.*

(i) *R has at most rank one.*

(ii) *For every $T \in \mathscr{B}(X)$ and all distinct scalars $\alpha, \beta \in \mathbb{C}$,*

$$\sigma(T + \alpha R) \cap \sigma(T + \beta R) \subseteq \sigma(T). \tag{1.2.5}$$

(iii) *Condition (1.2.5) is satisfied for every $T \in \mathscr{B}(X)$ of rank at most two.*

(iv) *For every $T \in \mathscr{B}(X)$, there exists a compact subset K_T of the complex plane, such that*

$$\sigma(T + \alpha R) \cap \sigma(T + \beta R) \subseteq K_T, \tag{1.2.6}$$

for all distinct scalars $\alpha, \beta \in \mathbb{C}$.

(v) *Condition (1.2.6) is satisfied for every $T \in \mathscr{B}(X)$ of rank at most two.*

Another big achievement in attempts to answer Kaplansky's question was done by Aupetit in [8]. He showed that Kaplansky's conjecture holds true if $\mathscr{A}$ and $\mathscr{B}$ are von Neumann algebras.

Theorem 1.2.13 (Aupetit [8], 2000)**.** *Let $\mathscr{A}$ and $\mathscr{B}$ be two von Neumann algebras. Then every surjective linear map $\varphi : \mathscr{A} \to \mathscr{B}$ that preserves the spectrum is a Jordan isomorphism.*

A similar result is obtained for Jordan-von Neumann algebras. As in the case of Banach algebras of all bounded linear operators, in this case, the problem is reduced to a characterization of linear maps preserving idempotents. A spectral characterization of idempotent elements is obtained.

Theorem 1.2.14 (Aupetit [8], 2000)**.** *Let $\mathscr{A}$ be a semisimple Banach (or Jordan-Banach) algebra. The following properties are equivalent.*

(i) *a is an idempotent element of $\mathscr{A}$.*

(ii) *$\sigma(a) \subset \{0, 1\}$ and there exist $r, C > 0$ such that*

$$\sigma(x) \subset \sigma(a) + C\|x - a\|, \qquad \textit{for} \quad \|x - a\| < r.$$

With the aid of this characterization it is concluded that a spectrum-preserving surjective linear map φ transforms a set of orthogonal idempotents of $\mathscr{A}$ into a set of orthogonal idempotents of $\mathscr{B}$. And after, to prove the Theorem 1.2.13, Aupetit uses the fact that in a von-Neumann algebra every self-adjoint element is the limit of a sequence of linear combinations of orthogonal idempotents; see [73, Proposition 1.3.1 and Lemma 1.7.5]. The same fact is true for Jordan-von Neumann algebras; see [38, Proposition 4.2.3], and the proof of the Theorem 1.2.13 runs almost the same way for this case.

Another interesting and important result belongs to Brešar, Fošner and Šemrl [25]. They have showed that Kaplansky's conjecture holds true if $\mathscr{A}$ and $\mathscr{B}$ are semisimple Banach algebras and if $\mathscr{A}$ has a big socle.

Theorem 1.2.15 (Brešar-Fošner-Šemrl [25], 2003)**.** *Let $\mathscr{A}$ and $\mathscr{B}$ be semisimple Banach algebras and let $\varphi : \mathscr{A} \to \mathscr{B}$ be a unital bijective linear operator that preserves invertibility. Then*

$$\varphi^{-1}(\varphi(a^2) - (\varphi(a))^2) \cdot Soc(\mathscr{A}) = 0 \qquad (a \in \mathscr{A}).$$

In particular, if $Soc(\mathscr{A})$ is an essential ideal of $\mathscr{A}$, then φ is a Jordan isomorphism.

In a primitive algebra every nonzero ideal is essential and as a primitive algebra is prime, so, using Herstein's result [44] about Jordan homomorphisms, we obtain the following corollary, which is a generalization of Sourour's Theorem 1.2.11 from [79].

Corollary 1.2.16 (Brešar-Fošner-Šemrl [25], 2003)**.** *Let $\mathscr{A}$ be a primitive Banach algebra with nonzero socle, and let $\mathscr{B}$ be a semisimple Banach algebra. If $\varphi : \mathscr{A} \to \mathscr{B}$ is a unital bijective linear operator that preserves invertibility, then φ is either an isomorphism or anti-isomorphism.*

1.3 Preservers of different moduli of operators

As two chapters of this thesis are devoted to the preservers of moduli of operators, we discuss in this section the important results obtained in this specific case of preservers. Let $\mathrm{d}(\cdot)$ stand for any of the spectral quantities $\mathrm{m}(\cdot)$, $\mathrm{q}(\cdot)$, $\mathrm{M}(\cdot)$ and $\gamma(\cdot)$. We say that a map $\varphi : \mathscr{A} \to \mathscr{B}$ preserves the spectral quantity $\mathrm{d}(\cdot)$ if

$$\mathrm{d}(\varphi(x)) = \mathrm{d}(x), \qquad (x \in \mathscr{A}).$$

First steps have been done by Mbekhta in [65, 66]. In [65] he has characterized surjective unital linear maps $\varphi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ that preserve the reduced minimum modulus.

Theorem 1.3.1 (Mbekhta [65], 2007)**.** *If $\varphi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ is a surjective unital linear map, then $\gamma(\varphi(T)) = \gamma(T)$ for every $T \in \mathscr{B}(\mathscr{H})$ if and only if there exists a unitary operator $U \in \mathscr{B}(\mathscr{H})$ such that φ takes one of the following forms*

$$\varphi(T) = UTU^* \quad \textit{or} \quad \varphi(T) = UT^{tr}U^*, \qquad (T \in \mathscr{B}(\mathscr{H})),$$

where T^{tr} denotes the transpose of T with respect to an arbitrary but a fixed orthonormal basis in $\mathscr{H}$.

Then he obtained a similar result for the preservers of the minimum or the surjectivity modulus in [66].

Theorem 1.3.2 (Mbekhta [66], 2010)**.** *Let $\varphi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ be a surjective unital linear map. Then the following conditions are equivalent.*

(i) $m(\varphi(T)) = m(T)$ *for every* $T \in \mathscr{B}(\mathscr{H})$.

(ii) $q(\varphi(T)) = q(T)$ *for every* $T \in \mathscr{B}(\mathscr{H})$.

(iii) There exists a unitary operator $U \in \mathscr{B}(\mathscr{H})$ such that

$$\varphi(T) = UTU^*, \qquad (T \in \mathscr{B}(\mathscr{H})).$$

After these papers Mbekhta stated a conjecture that these results are true without the assumption of φ being unital. In [15] Bourhim, Burgos and Shulman have proved this conjecture for the more general case of C^*-algebras.

Theorem 1.3.3 (Bourhim-Burgos-Shulman [15], 2010)**.** *Let $\mathscr{A}$ be a semisimple Banach algebra and let $\mathscr{B}$ be a C^*-algebra. If $\varphi : \mathscr{A} \to \mathscr{B}$ is a surjective linear map for which*

$$d(\varphi(x)) = d(x), \qquad (x \in \mathscr{A}),$$

then $\mathscr{A}$ (for its norm and some involution) is a C^*-algebra, and φ is an isometric Jordan *-isomorphism multiplied by a unitary element of $\mathscr{B}$.

Let us make couple of comments about this theorem. As we know from Kadison's paper [54], a surjective linear map between two C^*-algebras $\mathscr{A}$ and $\mathscr{B}$ is an isometry if and only if it is a Jordan *-isomorphism multiplied by a unitary element in $\mathscr{B}$, so we can reformulate the statement of Theorem 1.3.3 and say that it claims that the map φ is an isometry. Also we want to mention that the role of $\mathscr{A}$ or $\mathscr{B}$ being a C^*-algebra, is symmetrical since it is proved in [15] that a surjective linear map $\varphi : \mathscr{A} \to \mathscr{B}$ between Banach algebras is bijective, provided it preserves the spectral quantity $\mathrm{d}(\cdot)$.

In [15] another result for C^*-algebras $\mathscr{A}$ and $\mathscr{B}$ is obtained, where a linear map $\varphi : \mathscr{A} \to \mathscr{B}$ is supposed unital instead of being surjective and, of course, it is supposed to preserve the spectral quantity $\mathrm{d}(\cdot)$.

Theorem 1.3.4 (Bourhim-Burgos-Shulman [15], 2010)**.** *Let $\mathscr{A}$ and $\mathscr{B}$ be C^*-algebras. If $\varphi : \mathscr{A} \to \mathscr{B}$ is a unital linear map such that $d(x) = d(\varphi(x))$ for all $x \in \mathscr{A}$, then φ is an isometric Jordan *-homomorphism.*

It should be noted that the converses of these theorems are not true. Before giving an example, let us first note that every unital anti-homomorphism $\varphi : \mathscr{A} \to \mathscr{B}$ between C^*-algebras $\mathscr{A}$ and $\mathscr{B}$ satisfies $\sigma(xx^*) = \sigma(\varphi(xx^*)) = \sigma(\varphi(x^*)\varphi(x))$, so, by (1.1.3), $\mathrm{m}(\varphi(x)) = \mathrm{q}(x)$ and $\mathrm{q}(\varphi x)) = \mathrm{m}(x)$, hence $\mathrm{M}(\varphi(x)) = \mathrm{M}(x)$ and, by (1.1.4), $\gamma(\varphi(x)) = \gamma(x)$ for every $x \in \mathscr{A}$. Now we define the linear map $\varphi : T \mapsto T^{tr}$ on $\mathscr{B}(\mathscr{H})$, where T^{tr} is the transpose of T with respect to an arbitrary but a fixed orthonormal basis in an infinite-dimensional complex Hilbert space $\mathscr{H}$. We can see that φ is an isometric Jordan *-isomorphism, but it does not preserve neither the minimum modulus nor the surjectivity modulus, although it preserves the maximum modulus and the reduced minimum modulus. However, we note that if either in $\mathscr{A}$ or in $\mathscr{B}$ every left (right) invertible element is invertible, particularly if $\mathscr{H}$ is finite-dimensional, then the map φ preserves also the minimum modulus and the surjectivity modulus.

The main tool to prove these results is the following lemma, where the characterization of hermitian elements in a Banach algebra in terms of the minimum, surjectivity, maximum and reduced minimum moduli is given.

Lemma 1.3.5 (Bourhim-Bourgos-Shulman [15], 2010)**.** *Let $\mathscr{A}$ be an element of a Banach algebra $\mathscr{A}$. The following assertions are equivalent.*

(i) *a is hermitian.*

(ii) $\|\mathbf{1} + ita\| = 1 + o(t)$ *as* $t \to \infty$.

(iii) $d(\mathbf{1} + ita) = 1 + o(t)$ *as* $t \to \infty$.

(iv) $d(\mathbf{1} + ita) \geq 1 + o(t)$ *as* $t \to \infty$.

The last theorem that we are going to cite in this section is the result of Bourhim from [13], where the characterization of additive surjective maps between Banach algebras $\mathscr{B}(X)$ and $\mathscr{B}(Y)$ that preserve any of the minimum, surjectivity, maximum or reduced minimum moduli is obtained.

Theorem 1.3.6 (Bourhim [13], 2012)**.** *An additive surjective map $\varphi : \mathscr{B}(X) \to \mathscr{B}(Y)$ preserves the spectral quantity $d(\cdot)$ if and only if φ has one of the following forms.*

(i) *There are bijective isometries $U : X \to Y$ and $V : Y \to X$ both linear or both conjugate linear such that*

$$\varphi(T) = UTV, \qquad (T \in \mathscr{B}(X)).$$

(ii) *There are bijective isometries $U : X^* \to Y$ and $V : Y \to X^*$ both linear or both conjugate linear such that*

$$\varphi(T) = UT^*V, \qquad (T \in \mathscr{B}(X)).$$

This case can occur only if both X and Y are reflexive.

We note that the same result for the minimum modulus and the surjectivity modulus was proved also by Mbekhta-Oudghiri in [67].

Chapter 2

New approaches

2.1 Nonlinear maps preserving the minimum and surjectivity moduli

In this chapter we will discuss the results obtained in the thesis, the motivations for their consideration and we will provide short schemes of the proofs of main theorems. This section is about the Chapter 3.

The preserver problems that have been considered in the previous chapter had this general formulation: Find a characterization of a surjective map $\varphi : \mathscr{A} \to \mathscr{B}$ between Banach semisimple algebras $\mathscr{A}$ and $\mathscr{B}$ if for every element a in $\mathscr{A}$ a has some property if (and only if) $\varphi(a)$ has it. There is also another type of a problem, where it is supposed that $a - b$ has some property if (and only if) $\varphi(a) - \varphi(b)$ has it, for every $a, b \in \mathscr{A}$. Chapter 3 and Chapter 4 are devoted to this kind of nonlinear preserver problems between Banach algebras $\mathscr{B}(X)$ and $\mathscr{B}(Y)$.

One of the first results for such a problem was obtained in [57] by Kowalski, and Słodkowski. They have extended Gleason-Kahane-Żelazko theorem as follows.

Theorem 2.1.1 (Kowalski-Słodkowski [57], 1980)**.** *Let a functional $f : \mathscr{A} \to \mathbb{C}$ satisfy $f(0) = 0$ and*

$$f(x) - f(y) \in \sigma(x - y), \qquad (x, y \in \mathscr{A}).$$

Then f is linear and multiplicative.

Inspired by this in [43] Havlicek and Šemrl asked themselves to find the characterization of bijective maps φ that satisfy the condition

$$\varphi(S) - \varphi(T) \text{ is invertible} \iff S - T \text{ is invertible} \tag{2.1.1}$$

for matrix or operator algebras. In fact for the finite-dimensional case they have obtained more general results for spaces of rectangular matrices. Instead of saying that a matrix is invertible, we will say that it is of full rank; this will also be convenient for rectangular matrices. We denote by

$M_{m\times n}(\mathbb{F})$ the vector space of matrices over an arbitrary field $\mathbb{F}$ and $T_\sigma := [\sigma(t_{ij})]$, where $T = [t_{ij}] \in M_{m\times n}(\mathbb{F})$ and σ is an automorphism of the field $\mathbb{F}$.

Theorem 2.1.2 (Havlicek-Šemrl [43], 2006)**.** *Let $\mathbb{F}$ be a field with at least three elements and m, n integers with $m \geq n \geq 2$. Assume that $\varphi : M_{m\times n}(\mathbb{F}) \to M_{m\times n}(\mathbb{F})$ is a bijective map such that for every $S, T \in M_{m\times n}(\mathbb{F})$ we have*

$$\varphi(S) - \varphi(T) \text{ is of full rank} \iff S - T \text{ is of full rank.}$$

Then there exist an invertible matrix $A \in M_m(\mathbb{F})$, an invertible matrix $B \in M_n(\mathbb{F})$, a matrix $R \in M_{m\times n}(\mathbb{F})$ and an automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ such that

$$\varphi(T) = AT_\sigma B + R, \qquad (T \in M_{m\times n}(\mathbb{F})).$$

If $m = n$, then we have the additional possibility that

$$\varphi(T) = A(T_\sigma)^{tr} B + R, \qquad (T \in M_n(\mathbb{F})),$$

where $A, B, R \in M_n(\mathbb{F})$ with A and B invertible and σ is an automorphism of $\mathbb{F}$.

We see that in the finite-dimensional case the condition (2.1.1) has implied the semilinearity of the maps up to a translation, while in the infinite-dimensional case the maps are proved to be linear or conjugate-linear up to a translation, as we see in the following theorem, where the characterization of bijective maps $\varphi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ that satisfy (2.1.1) is given.

Theorem 2.1.3 (Havlicek-Šemrl [43], 2006)**.** *Let $\mathscr{H}$ be a Hilbert space and $\varphi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ be a bijective map such that for every pair $S, T \in \mathscr{B}(\mathscr{H})$ the operator $S - T$ is invertible if and only if $\varphi(S) - \varphi(T)$ is invertible. Then there exist $R \in \mathscr{B}(\mathscr{H})$ and invertible $A, B \in \mathscr{B}(\mathscr{H})$ such that*

$$\varphi(T) = AT^\# B + R, \qquad (T \in \mathscr{B}(\mathscr{H})),$$

where $T^\#$ stands for T, or T^{tr}, the transpose with respect to an arbitrary but fixed orthonormal basis, or T^*, the usual adjoint of T in the Hilbert space sense, or $(T^{tr})^*$.

In[48] Hou and Huang extended this result to the case of maps φ between Banach algebras $\mathscr{B}(X)$ and $\mathscr{B}(Y)$.

Theorem 2.1.4 (Hou-Huang [48], 2009)**.** *Let X, Y be Banach spaces and $\varphi : \mathscr{B}(X) \to \mathscr{B}(Y)$ be a bijective map such that for every pair $S, T \in \mathscr{B}(X)$ the operator $S - T$ is invertible if and only if $\varphi(S) - \varphi(T)$ is invertible. Then one of the following situations holds:*

(i) There is an operator $R \in \mathscr{B}(Y)$ and there are bijective continuous mappings $A : X \to Y$ and $B : Y \to X$, either both linear or both conjugate linear, such that

$$\varphi(T) = ATB + R, \qquad (T \in \mathscr{B}(X)).$$

(ii) *There is an operator $R \in \mathscr{B}(Y)$ and there are bijective continuous mappings $A : X^* \to Y$ and $B : Y \to X^*$, either both linear or both conjugate linear, such that*

$$\varphi(T) = AT^*B + R, \qquad (T \in \mathscr{B}(X)).$$

This case may occur only if both X and Y are reflexive.

The main tool to prove all these mentioned theorems is the characterization of adjacent operators. Recall that two operators (matrices) S and T are said to be *adjacent* if $S - T$ is a rank one operator (matrix). We will cite here the result of Hou and Huang for the more general case of Banach algebras $\mathscr{B}(X)$. They have used the same proof of Havlicek and Šemrl, who had obtained the same characterization for the Banach algebra $\mathscr{B}(\mathscr{H})$ and the spaces $\mathscr{M}_{m\times n}(\mathbb{F})$.

Lemma 2.1.5 (Hou-Huang [48], 2009)**.** *For two different operators S and T in $\mathscr{B}(X)$, the following statements are equivalent.*

(i) *S and T are adjacent.*

(ii) *There exists $R \in \mathscr{B}(X)\backslash\{S, T\}$ such that for every $N \in \mathscr{B}(X)$, $N - R$ is invertible yields that $N - S$ or $N - T$ is invertible.*

Every rank one operator is adjacent to zero, every rank two operator is adjacent to some rank one operator, etc. So, if φ is a surjective map satisfying (2.1.1), then, by this lemma, we conclude that φ maps the ideal of all finite rank operators $\mathscr{F}(X)$ onto the ideal of all finite rank operators $\mathscr{F}(Y)$. Another crucial role in the proofs of the Theorem 2.1.3 and the Theorem 2.1.4 belongs to the following result from [70].

Lemma 2.1.6 (Petek-Šemrl [70], 2002)**.** *Assume that X and Y are Banach spaces of dimensions at least 2, and let φ be a bijective map from $\mathscr{F}(X)$ into $\mathscr{F}(Y)$ such that whenever S, T are operators in $\mathscr{F}(X)$ then one has*

$$\varphi(S) - \varphi(T) \text{ has rank one} \iff S - T \text{ has rank one.}$$

Then one of the following situations hold:

(i) *There are an automorphism σ of $\mathbb{C}$, $R \in \mathscr{B}(Y)$, and bijective σ-semilinear maps $A : X \to Y$ and $B : X^* \to Y^*$ such that $T \mapsto \varphi(T) - R$ is an additive map defined by*

$$\varphi(x \otimes f) - R = Ax \otimes Bf, \qquad (x \in X,\ f \in X^*).$$

(ii) *There are an automorphism σ of $\mathbb{C}$, $R \in \mathscr{B}(Y)$, and bijective σ-semilinear maps $A : X \to Y^*$ and $B : X^* \to Y$ such that $T \mapsto \varphi(T) - R$ is an additive map defined by*

$$\varphi(x \otimes f) - R = Bf \otimes Ax, \qquad (x \in X,\ f \in X^*).$$

Inspired by these results, we asked ourselves what would be the characterization of surjective maps $\varphi : \mathscr{B}(X) \to \mathscr{B}(Y)$ that satisfy one of the following conditions:

$$\mathrm{c}(\varphi(S) - \varphi(T)) = \mathrm{c}(S - T), \qquad (S, T \in \mathscr{B}(X)), \tag{2.1.2}$$

or

$$\mathrm{c}(\varphi(S) + \varphi(T)) = \mathrm{c}(S + T), \qquad (S, T \in \mathscr{B}(X)), \tag{2.1.3}$$

where $\mathrm{c}(\cdot)$ denotes either the minimum modulus, or the surjectivity modulus, or the maximum modulus. The Chapter 3 of this thesis is devoted to this problem and here is the main result that we have obtained.

Theorem 2.1.7 (Bourhim-Mashreghi-Stepanyan [20], 2014)**.** *Assume that φ is a surjective map from $\mathscr{B}(X)$ to $\mathscr{B}(Y)$ satisfying (2.1.2). Then one of the following situations hold:*

(i) There are $R \in \mathscr{B}(Y)$ and bijective isometries $U : X \to Y$ and $V : Y \to X$, either both linear or both conjugate linear, such that

$$\varphi(T) = UTV + R, \qquad (T \in \mathscr{B}(X)).$$

(ii) There are bijective isometries $U : X^* \to Y$ and $V : Y \to X^*$, either both linear or both conjugate linear, such that

$$\varphi(T) = UT^*V + R, \qquad (T \in \mathscr{B}(X)).$$

This case can occur only if both X and Y are reflexive.

If a surjective map $\varphi : \mathscr{B}(X) \to \mathscr{B}(Y)$ satisfies (2.1.3), then we have the same conclusion with $R = 0$ in the forms.

If we remember the facts about moduli, that have been discussed in the first chapter, then we can say that if φ satisfies (2.1.2), then in particular it satisfies

$$S - T \in \Omega(X) \iff \varphi(S) - \varphi(T) \in \Omega(X) \tag{2.1.4}$$

where $\Omega(X)$ stands either for $\Omega_{\text{LB}}(X)$, or $\Omega_{\text{Surj}}(X)$, or $\Omega_{\text{LB-or-Surj}}(X)$. And likewise, if φ satisfies (2.1.3), then it also satisfies

$$S + T \in \Omega(X) \iff \varphi(S) + \varphi(T) \in \Omega(X).$$

To prove Theorem 2.1.7, we have shown that a result, similar to Lemma 2.1.5, holds true for our cases. The proof of this lemma is done in a constructive way.

Lemma 2.1.8 (Bourhim-Mashreghi-Stepanyan [20], 2014)**.** *For two different operators S and T in $\mathscr{B}(X)$, the following statements are equivalent.*

(i) S and T are adjacent.

(ii) There is an operator $R \in \mathscr{B}(X)\backslash\{S,T\}$ such that either $N-S \in \Omega(X)$ or $N-T \in \Omega(X)$ for all $N \in \Omega(X)+R$.

Now we can give a short sketch of the proof of Theorem 2.1.7. Let $\varphi : \mathscr{B}(X) \to \mathscr{B}(Y)$ be a surjective map that satisfies the condition (2.1.2). First we prove that φ is injective, then by Lemma 2.1.8 we conclude that φ maps the ideal of all finite rank operators $\mathscr{F}(X)$ onto the ideal of all finite rank operators $\mathscr{F}(Y)$, and so, as in [43] and [48], we arrive at a point, where we can use Lemma 2.1.6 and get the forms of φ on rank one operators. With these forms, we prove that $\varphi(\mathbf{1})$ is invertible. For a fixed invertible operator $T_0 \in \mathscr{B}(X)$, we define a map $\phi(T) := \varphi(T_0 T)$, which also satisfies the conditions of Theorem 2.1.7, so $\phi(\mathbf{1}) = \varphi(T_0)$ is invertible and we conclude that φ preserves invertibility. As φ^{-1} also satisfies the conditions of Theorem 2.1.7, so, in fact, φ preserves invertibility in both directions. With another simple substitution $\psi(S) := \varphi(S+T) - \varphi(T)$ for an arbitrary fixed operator $T \in \mathscr{B}(X)$, which also satisfies the conditions of Theorem 2.1.7, we conclude that $S-T$ is invertible if and only if $\psi(S-T) = \varphi(S) - \varphi(T)$ if invertible, hence we see that the conditions of Hou and Huang's Theorem 2.1.4 are satisfied, and we have the forms of φ mentioned therein. The last step to obtain the exact same forms of our theorem is done by noticing the following equality

$$\|T\|^{-1} = \mathrm{c}\left(T^{-1}\right) = \mathrm{c}\left(\varphi(T^{-1}) - \varphi(0)\right) = \mathrm{c}\left(AT^{-1}B\right) = \|B^{-1}TA^{-1}\|^{-1},$$

for an invertible operator $T \in \mathscr{B}(X)$ and using the following lemma cited from [13].

Lemma 2.1.9. *For two bijective transformations $A \in \mathscr{B}(X,Y)$ and $B \in \mathscr{B}(Y,X)$, the following statements are equivalent.*

(i) $\|ATB\| = \|T\|$ for all invertible operators $T \in \mathscr{B}(X)$.

(ii) A and B are isometries multiplied by complex scalars λ and μ such that $\lambda\mu = 1$.

The proof for the case when a surjective map $\varphi : \mathscr{B}(X) \to \mathscr{B}(Y)$ satisfies (2.1.3) is proved the same way. The Lemma 2.1.8 and the Theorem 2.1.7 are proved also for the cases when as $\Omega(X)$ is taken the subset of all injective operators in $\mathscr{B}(X)$, noted by $\Omega_{\text{Inj}}(X)$, or the subset of all injective or surjective operators in $\mathscr{B}(X)$, noted by $\Omega_{\text{Inj-or-Surj}}(X)$. And at the end of the discussion of the Chapter 3, we would like to mention that the finite-dimensional case of matrix algebra $M_n(\mathbb{C})$ is also considered. As a finite-dimensional operator (a matrix) is injective or surjective if and only if it is invertible, then the condition $\mathrm{c}(\varphi(S) - \varphi(T)) = \mathrm{c}(S-T)$ for all $S,T \in M_n(\mathbb{C})$ is equivalent to (2.1.1) for all $S,T \in M_n(\mathbb{C})$, so it is possible to use Havlicek and Šemrl's Theorem 2.1.2, but in this case we obtain additionally that the automorphism $\sigma : \mathbb{C} \to \mathbb{C}$ must be the identity or conjugation map.

2.2 Nonlinear maps preserving the reduced minimum modulus of operators

The natural extension of the results described in the previous section would be characterizing surjective maps $\varphi : \mathscr{B}(X) \to \mathscr{B}(Y)$ that satisfy

$$\gamma(\varphi(S) - \varphi(T)) = \gamma(S - T), \qquad (S, T \in \mathscr{B}(X)), \tag{2.2.5}$$

or

$$\gamma(\varphi(S) + \varphi(T)) = \gamma(S + T), \qquad (S, T \in \mathscr{B}(X)). \tag{2.2.6}$$

Chapter 4 is devoted to this problem. Before stating the main result let us note that if φ satisfies (2.2.6) then it automatically satisfies (2.2.5), because in this case $\varphi(-T) = -\varphi(T)$ for all $T \in \mathscr{B}(X)$, as $\gamma(\varphi(-T) + \varphi(T)) = \gamma(-T + T) = \infty$. And also, if φ satisfies (2.2.5), then it is injective, as when $\varphi(S) = \varphi(T)$, then $\infty = \gamma(\varphi(S) - \varphi(T)) = \gamma(S - T)$, hence $S = T$. So we can assume that our map φ is bijective. Our result is obtained for the case, when φ is additionally supposed to be bicontinuous.

Theorem 2.2.1 (Mashreghi-Stepanyan [63], 2016)**.** *A bicontinuous bijective map φ from $\mathscr{B}(X)$ into $\mathscr{B}(Y)$ satisfies (2.2.5) if and only if one of the following situations hold:*

(i) There are $R \in \mathscr{B}(Y)$ and bijective isometries $U : X \to Y$ and $V : Y \to X$, either both linear or both conjugate linear, such that

$$\varphi(T) = UTV + R, \qquad (T \in \mathscr{B}(X)).$$

(ii) There are $R \in \mathscr{B}(Y)$ and bijective isometries $U : X^* \to Y$ and $V : Y \to X^*$, either both linear or both conjugate linear, such that

$$\varphi(T) = UT^*V + R, \qquad (T \in \mathscr{B}(X)).$$

This case can occur only if both X and Y are reflexive.

If a bijective bicontinuous map $\varphi : \mathscr{B}(X) \to \mathscr{B}(Y)$ satisfies (2.2.6), then we have the same conclusion with $R = 0$ in the forms.

Here is the short sketch of the proof of this theorem. We remark that $\varphi - \varphi(0)$ also satisfies the conditions of the theorem, so, without loss of generality, we may assume that $\varphi(0) = 0$. The first and important step is to show that

$$\mathrm{M}(\varphi(T)) > 0 \iff \mathrm{M}(T) > 0, \qquad (T \in \mathscr{B}(X)). \tag{2.2.7}$$

We used an idea from [13] to prove this equivalence. We take an operator $T_0 \in \mathscr{B}(X)$ such that $\mathrm{M}(T_0) > 0$ and assuming that $\mathrm{M}(\varphi(T_0)) = 0$, we show that there exist $x \in X$ and $f \in X^*$ such that

$$r \geq \delta\gamma(T_0 - \varphi^{-1}(rx \otimes f)) = \delta\mathrm{M}(T_0 - \varphi^{-1}(rx \otimes f))$$

for all sufficiently small numbers r. As we have supposed that φ is bicontinuous, so, by the continuousness of the maximum modulus, we obtain that the right side tends to $\mathrm{M}(T_0) > 0$ as r goes to zero, hence we get a contradiction.

As the next step, we define a map $\psi(S) := \varphi(S+T) - \varphi(T)$, for an arbitrary fixed $T \in \mathscr{B}(X)$. ψ also satisfies the conditions of Theorem 2.2.1, hence (2.2.7) holds true for it and by replacing S by $S - T$ we get

$$\mathrm{M}(\psi(S-T)) = \mathrm{M}(\varphi(S) - \varphi(T)) > 0 \iff \mathrm{M}(S-T) > 0$$

for all $S, T \in \mathscr{B}(X)$. Taking into consideration the fact, that when the maximum modulus of an operator is positive, then it is equal to the reduced minimum modulus of this operator and the condition (2.2.5), we get

$$\mathrm{M}(\varphi(S) - \varphi(T)) = \mathrm{M}(S-T), \qquad (S, T \in \mathscr{B}(X)).$$

From here, by Theorem 2.1.7, we obtain the forms for φ, where $R = \varphi(0)$. The last part of the theorem follows easily from the remark, that $\varphi(0)$ is necessarily 0, when φ satisfies (2.2.6).

We want to finish the discussion of the Chapter 4 by noting that the finite dimensional case is also obtained. The proof is similar to this one, it uses the results for the finite dimensional case obtained in [20].

2.3 Nonlinear maps between Banach algebras preserving the spectrum

In this section we will discuss the Chapter 5. It is devoted to another kind of nonlinear preserver problem. The question is to describe surjective maps φ between semisimple Banach algebras $\mathscr{A}$ and $\mathscr{B}$ that satisfy the following condition

$$\sigma\big(\varphi(a)\varphi(b)\big) = \sigma(ab), \qquad (a, b \in \mathscr{A}). \tag{2.3.8}$$

The motivation was the following result of Molnár [69].

Theorem 2.3.1 (Molnár [69], 2001)**.** *Let $\mathscr{H}$ be a Hilbert space. A surjective map $\varphi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ satisfies*

$$\sigma\big(\varphi(T)\varphi(S)\big) = \sigma(TS), \qquad (T, S \in \mathscr{B}(\mathscr{H})),$$

if and only if there exists an invertible operator $A \in \mathscr{B}(\mathscr{H})$ such that either

$$\varphi(T) = ATA^{-1}, \qquad (T \in \mathscr{B}(\mathscr{H})),$$

or

$$\varphi(T) = -ATA^{-1}, \qquad (T \in \mathscr{B}(\mathscr{H})).$$

There have been many extensions of this result; see for instance [1, 27, 40–42, 45, 47, 49, 53, 58–61, 68, 71, 72, 80, 81].

In the Chapter 5 we have answered the question (2.3.8) assuming additionally that $\mathscr{B}$ has an essential socle and in a more general way, considering maps φ_1 and φ_2.

Theorem 2.3.2 (Bourhim-Mashreghi-Stepanyan)**.** *Assume that $\mathscr{A}$ is semisimple and $\mathscr{B}$ has an essential socle. If surjective maps $\varphi_1, \varphi_2 : \mathscr{A} \to \mathscr{B}$ satisfy*

$$\sigma\big(\varphi_1(a)\varphi_2(b)\big) = \sigma(ab), \qquad (a, b \in \mathscr{A}), \tag{2.3.9}$$

then the maps $\varphi_1\varphi_2(\mathbf{1})$ and $\varphi_1(\mathbf{1})\varphi_2$ coincide and are Jordan isomorphisms.

We prove lemmas, which say that if surjective maps $\varphi_1, \varphi_2 : \mathscr{A} \to \mathscr{B}$ satisfy (2.3.9), then

$$\varphi_1(\mathrm{rad}(\mathscr{A})) = \varphi_2(\mathrm{rad}(\mathscr{A})) = \mathrm{rad}(\mathscr{B}),$$

$$\varphi_1\big(\mathscr{F}_1(\mathscr{A})\big) = \varphi_2\big(\mathscr{F}_1(\mathscr{A})\big) = \mathscr{F}_1(\mathscr{B}) \qquad \text{and} \qquad \varphi_1\big(\mathrm{Soc}(\mathscr{A})\big) = \varphi_2\big(\mathrm{Soc}(\mathscr{A})\big) = \mathrm{Soc}(\mathscr{B}), \tag{2.3.10}$$

so, in fact, if the conditions of the Theorem 2.3.2 are satisfied, then both Banach algebras $\mathscr{A}$ and $\mathscr{B}$ are semisimple and have essential socles. An immediate corollary of the Theorem 2.3.2 is the case when $\varphi_1 = \varphi_2$.

Corollary 2.3.3. *Assume that $\mathscr{A}$ is semisimple and $\mathscr{B}$ has an essential socle. If a surjective map $\varphi : \mathscr{A} \to \mathscr{B}$ satisfies (2.3.8), then $\varphi(\mathbf{1})$ is a central invertible element of $\mathscr{B}$ for which $\varphi(\mathbf{1})^2 = \mathbf{1}$ and $\varphi(\mathbf{1})\varphi$ is a Jordan isomorphism.*

And, taking into consideration that a Banach algebra $\mathscr{B}(X)$ is semisimple and has an essential socle, we obtain another corollary.

Corollary 2.3.4. *Two surjective maps $\varphi_1, \varphi_2 : \mathscr{B}(X) \to \mathscr{B}(Y)$ satisfy (2.3.9) if and only if one of the following statements holds.*

- *(i) There exist two bijective mappings $A,\ B \in \mathscr{B}(X, Y)$ such that $\varphi_1(T) = ATB^{-1}$ and $\varphi_2(T) = BTA^{-1}$ for all $T \in \mathscr{B}(X)$.*
- *(ii) There exist two bijective mappings $A,\ B \in \mathscr{B}(X^*, Y)$ such that $\varphi_1(T) = AT^*B^{-1}$ and $\varphi_2(T) = BT^*A^{-1}$ for all $T \in \mathscr{B}(X)$. This case can not occur if X or Y is not reflexive or if there is a left invertible operator in $\mathscr{B}(X)$ which is not invertible.*

We omit the details of the proof of the Theorem 2.3.2. We just want to mention, that it is inspired and uses ideas from [9, 25], where it is shown, that there is a function τ on $\mathscr{F}_1(\mathscr{A})$ such that $\sigma(u) = \{0, \tau(u)\}$ for all $u \in \mathscr{F}_1(\mathscr{A})$ and that

$$\tau(au + bu) = \tau(au) + \tau(bu)$$

for all $a,\ b \in \mathscr{A}$ and $u \in \mathscr{F}_1(\mathscr{A})$. This equality helps to prove (2.3.10) and after the linearity of φ_1 and φ_2. And we note that the fact of $\varphi_1\varphi_2(\mathbf{1})$ and $\varphi_1(\mathbf{1})\varphi_2$ being Jordan isomorphisms, is followed from the main result of [25], which is the Theorem 1.2.15.

Chapter 3

Nonlinear maps preserving the minimum and surjectivity moduli

Résumé

Soit X et Y des espaces de Banach complexes de dimension infine et soit $\mathscr{B}(X)$ (resp. $\mathscr{B}(Y)$) l'algèbre de tous les opérateurs linéaires et bornés sur X (resp. sur Y). Nous décrivons des applications surjectives φ de $\mathscr{B}(X)$ sur $\mathscr{B}(Y)$, qui satisfont

$$\mathrm{c}(\varphi(S) \pm \varphi(T)) = \mathrm{c}(S \pm T)$$

pour tous $S,\ T \in \mathscr{B}(X)$, où $\mathrm{c}(\cdot)$ représente soit le module minimal, ou le module de surjectivité ou le module maximal. Nous obtenons aussi des résultats analogues pour le cas de dimension finie.

Abstract

Let X and Y be infinite-dimensional complex Banach spaces, and let $\mathscr{B}(X)$ (resp. $\mathscr{B}(Y)$) denote the algebra of all bounded linear operators on X (resp. on Y). We describe maps φ from $\mathscr{B}(X)$ onto $\mathscr{B}(Y)$ satisfying

$$\mathrm{c}(\varphi(S) \pm \varphi(T)) = \mathrm{c}(S \pm T)$$

for all $S,\ T \in \mathscr{B}(X)$, where $\mathrm{c}(\cdot)$ stands either for the minimum modulus, or the surjectivity modulus, or the maximum modulus. We also obtain analog results for the finite-dimensional case.

3.1 Introduction

There has been considerable interest in studying nonlinear maps on operators or matrices preserving the invertibility, the spectrum and its parts. In [11], Bhatia, Šemrl and Sourour characterized surjective maps preserving the spectral radius of the difference of matrices. In [69], Molnár studied maps preserving the spectrum of operator or matrix products and showed, in particular, that a surjective map φ on the algebra $\mathscr{B}(\mathscr{H})$ of all bounded linear operators on an infinite-dimensional complex Hilbert space $\mathscr{H}$ preserves the spectrum of operator products if and only if there exists an invertible operator $A \in \mathscr{B}(\mathscr{H})$ such that either $\varphi(T) = ATA^{-1}$ for all $T \in \mathscr{B}(\mathscr{H})$ or $\varphi(T) = -ATA^{-1}$ for all $T \in \mathscr{B}(\mathscr{H})$. His results have been extended in several direction for uniform algebras and semisimple commutative Banach algebras, and a number of results is obtained on maps preserving several spectral and local spectral quantities of operator or matrix product, or Jordan product, or Jordan triple product, or difference; see for instance [16–19, 27–30, 35, 40–43, 47, 49, 50, 53, 58, 60, 61, 68, 71, 72, 80] and the references therein.

Throughout this paper, X and Y denote infinite-dimensional complex Banach spaces, and $\mathscr{B}(X,Y)$ denotes the space of all bounded linear maps from X into Y. When $X = Y$, we simply write $\mathscr{B}(X)$ instead of $\mathscr{B}(X,X)$. The dual space of X will be denoted by X^*, and the Banach space adjoint of an operator $T \in \mathscr{B}(X)$ will be denoted by T^*. In [43], Havlicek and Šemrl gave a complete characterization of bijective maps φ on the algebra $\mathscr{B}(\mathscr{H})$ of all bounded linear operators on an infinite-dimensional complex Hilbert space $\mathscr{H}$ satisfying the condition (3.1.1). In [48], Hou and Huang characterized surjective maps between standard operator algebras on complex Banach spaces that completely preserve the spectrum or the invertibility in both directions. They also observed that Havlicek and Šemrl's result and its proof remains valid in the case of Banach spaces setting.

Theorem 3.1.1. *(Havlicek-Šemrl [43], Hou and Huang [48].) A map φ from $\mathscr{B}(X)$ onto $\mathscr{B}(Y)$ satisfies*

$$\varphi(S) - \varphi(T) \text{ is invertible} \iff S - T \text{ is invertible} \tag{3.1.1}$$

if and only if one of the following situations hold:

(i) There is an operator $R \in \mathscr{B}(Y)$ and there are bijective continuous mappings $A : X \to Y$ and $B : Y \to X$, either both linear or both conjugate linear, such that

$$\varphi(T) = ATB + R, \qquad T \in \mathscr{B}(X). \tag{3.1.2}$$

(ii) There is an operator $R \in \mathscr{B}(Y)$ and there are bijective continuous mappings $A : X^* \to Y$ and $B : Y \to X^*$, either both linear or both conjugate linear, such that

$$\varphi(T) = AT^*B + R, \qquad T \in \mathscr{B}(X). \tag{3.1.3}$$

This case may occur only if both X and Y are reflexive.

The *minimum modulus* of an operator $T \in \mathscr{B}(X)$ is $\mathrm{m}(T) := \inf\{\|Tx\| \colon x \in X,\ \|x\| = 1\}$, and is positive precisely when T is bounded below; i.e., T is injective and has a closed range. The *surjectivity modulus* of T is $\mathrm{q}(T) := \sup\{\varepsilon \geq 0 \colon \varepsilon B_X \subseteq T(B_X)\}$, and is positive if and only if T is surjective. Here, B_X is the closed unit ball of X. While, the *maximum modulus* of T is defined by $\mathrm{M}(T) := \max\{\mathrm{m}(T), \mathrm{q}(T)\}$, and is positive precisely when either T is bounded below or T is surjective. Note that $\mathrm{m}(T^*) = \mathrm{q}(T)$ and $\mathrm{q}(T^*) = \mathrm{m}(T)$ for all $T \in \mathscr{B}(X)$, and that

$$\mathrm{M}(T) = \mathrm{m}(T) = \mathrm{q}(T) = \|T^{-1}\|^{-1}$$

whenever T is invertible.

In this paper, we let c(.) stand for either the minimum modulus m(.), or the surjectivity modulus q(.), or the maximum modulus M(.). We establish a similar result to Theorem 3.1.1 of characterizing maps from $\mathscr{B}(X)$ onto $\mathscr{B}(Y)$ preserving any of the surjectivity, the injectivity, and the boundedness from below of the difference and sum of operators. We describe the adjacency of operators in term of any of the previous mentioned spectral quantities and use such a description to show that if a map φ from $\mathscr{B}(X)$ onto $\mathscr{B}(Y)$ satisfies (3.1.4), then φ satisfies (3.1.1) and thus Theorem 3.1.1 ensures that such a map φ takes either (3.1.2) or (3.1.3). Then we describe maps φ from $\mathscr{B}(X)$ onto $\mathscr{B}(Y)$ satisfying

$$\mathrm{c}(\varphi(S)-\varphi(T))=\mathrm{c}(S-T) \tag{3.1.4}$$

for all $S,\ T\in\mathscr{B}(X)$. We show that such a map φ is an isometry translated by an operator in $\mathscr{B}(Y)$. We also characterize maps φ from $\mathscr{B}(X)$ onto $\mathscr{B}(Y)$ satisfying

$$\mathrm{c}(\varphi(S)+\varphi(T))=\mathrm{c}(S+T) \tag{3.1.5}$$

for all $S,\ T\in\mathscr{B}(X)$. Furthermore, we obtain analog results for the finite dimensional case on maps preserving the minimum modulus of the difference of matrices. Our arguments are influenced by ideas from several papers [13, 15, 43, 48] and the approaches given therein, but the proof of our main results requires new ingredients such as the complete characterization of maps preserving the surjectivity and boundedness from below of the difference and sum of operators.

3.2 Preliminaries

In this section, we provide some essential lemmas that will be used in the proof of our main results. The first lemma is well known and its proof is quoted from [23] for the sake of completeness of the reader. Recall that the spectral radius of an operator $T\in\mathscr{B}(X)$ is $\mathrm{r}(T)=\lim_{n\to\infty}\|T^n\|^{\frac{1}{n}}$, and coincides with the maximum modulus of $\sigma(T)$, the spectrum of T. Given $x\in X$ and $f\in X^*$, we write $\langle x,f\rangle$ instead of $f(x)$ and $x\otimes f$ for the rank one operator defined by

$$(x\otimes f)(y):=\langle y,f\rangle x,\qquad (y\in X).$$

Lemma 3.2.1. *(Brešar-Šemrl [23].) Let $A\in\mathscr{B}(X)$. Then $A=0$ if and only if $\mathrm{r}(A+T)=0$ for all nilpotent operators $T\in\mathscr{B}(X)$ of rank at most one.*

Proof. Assume that $\mathrm{r}(A+T)=0$ for all nilpotent operators $T\in\mathscr{B}(X)$ of rank at most one. We suppose that $A\neq 0$ and obtain a contradiction.

Since $A\neq 0$, there is a vector x in X such that $Ax\neq 0$. Since $\mathrm{r}(A)=0$, the vectors Ax and x are linearly independent. Hence, there is a linear functional $f\in X^*$ such that $\langle x,f\rangle=\langle Ax,f\rangle=1$. Now, consider the rank one operator $T:=(-Ax+x)\otimes f$. It satisfies $T^2=0$ and $(T+A)x=x$, and the second identity implies $\mathrm{r}(A+T)\geq 1$, which is a contradiction. □

In passing, we would like to mention that inspecting the proof of Theorem 3.1.1, it is easy to notice that this result remains true when using sums in (3.1.1) instead of subtractions. But in such a case, one should have $R = 0$ in either form (3.1.2) or (3.1.3). Indeed, assume that φ is a map from $\mathscr{B}(X)$ onto $\mathscr{B}(Y)$ satisfying

$$\varphi(S) + \varphi(T) \text{ is invertible} \iff S + T \text{ is invertible,} \tag{3.2.6}$$

and let $\varphi(T_0) = 0$ for some $T_0 \in \mathscr{B}(X)$. Then for every $T \in \mathscr{B}(X)$, we have

$$\begin{aligned} T + T_0 \text{ is invertible} &\iff \phi(T) \text{ is invertible} \\ &\iff 2\phi(T) = \phi(T) + \phi(T) \text{ is invertible} \\ &\iff T + T \text{ is invertible} \\ &\iff T \text{ is invertible.} \end{aligned}$$

Replacing T by $T - \lambda\mathbf{1}$, one gets that $\sigma(T + T_0) = \sigma(T)$ for all $T \in \mathscr{B}(X)$. By Lemma 3.2.1, one has $T_0 = 0$ and $\varphi(0) = 0$.

This fact remains true if the invertibility in (3.2.6) is replaced by the surjectivity, boundedness from below, ect. In particular, the following lemma holds and shows that a map from $\mathscr{B}(X)$ onto $\mathscr{B}(Y)$ vanishes at zero provided that it satisfies (3.1.5).

Lemma 3.2.2. *If a map φ from $\mathscr{B}(X)$ onto $\mathscr{B}(Y)$ satisfies (3.1.5), then $\varphi(0) = 0$.*

The following lemma is quoted from [76, Theorem 3.1 and Corollary 3.2] and is also needed for the proof of our main results. The proofs presented therein are long and require several computations and applications of Hahn-Banach Theorem. However, a simple and shorter proof was given in [13].

Lemma 3.2.3. *(Skhiri [76].) For a bijective mapping $A \in \mathscr{B}(X, Y)$, the following statements are equivalent.*

(i) $\|ATA^{-1}\| = \|T\|$ *for all invertible operators* $T \in \mathscr{B}(X)$.

(ii) $\|ATA^{-1}\| \leq \|T\|$ *for all invertible operators* $T \in \mathscr{B}(X)$.

(iii) $\|ATA^{-1}\| \geq \|T\|$ *for all invertible operators* $T \in \mathscr{B}(X)$.

(iv) A is an isometry multiplied by a scalar.

3.3 Nonlinear maps preserving the minimum modulus of matrices

Let $M_n(\mathbb{C})$ be the algebra of all $n \times n$-complex matrices, and note that

$$\mathrm{m}(T) = \mathrm{q}(T) = \mathrm{M}(T)$$

for all matrices $T \in M_n(\mathbb{C})$. For an automorphism σ of $\mathbb{C}$ and a matrix $T = [t_{ij}] \in M_n(\mathbb{C})$, let

$$T_\sigma := [\sigma(t_{ij})].$$

Note that there are very large number of automorphisms of $\mathbb{C}$, but the identity and conjugation maps are the only continuous automorphisms; see [82]. On the other hand, it is well known that for any isometry σ on $\mathbb{C}$ there is a unimodular scalar α and a scalar $\beta \in \mathbb{C}$ such that either $\sigma(z) = \alpha z + \beta,\ (z \in \mathbb{C})$, or $\sigma(z) = \alpha\overline{z} + \beta,\ (z \in \mathbb{C})$. In particular, if such an isometry σ fixes 0 and 1, then σ is the identity or conjugation map.

In [43, Theorem 1.1], Havlicek and Šemrl obtained, in particular, a complete description of bijective maps φ on the algebra $M_n(\mathbb{F})$ of all $n \times n-$matrices over a field $\mathbb{F}$ with at least three elements that satisfy $\varphi(a) - \varphi(b)$ is invertible if and only if $a - b$ is invertible. For our purposes, we state the result by Havlicek and Šemrl only for the complex case and we refer the readers to their paper for the general result.

Theorem 3.3.1. *(Havlicek -Šemrl [43].) A surjective map φ on $M_n(\mathbb{C})$ satisfies*

$$\varphi(S) - \varphi(T) \text{ is invertible} \Longleftrightarrow S - T \text{ is invertible}$$

if and only if there are $A,\ B,\ R \in M_n(\mathbb{C})$, with A and B invertible matrices, and an automorphism σ of $\mathbb{C}$ such that either

$$\varphi(T) = AT_\sigma B + R, \qquad (T \in M_n(\mathbb{C})),$$

or

$$\varphi(T) = A(T_\sigma)^{tr} B + R, \qquad (T \in M_n(\mathbb{C})).$$

Now, we are in a position to state and prove the main result of this section that gives a complete description of surjective maps on $M_n(\mathbb{C})$ preserving the minimum modulus of the difference of matrices.

Theorem 3.3.2. *A surjective map φ on $M_n(\mathbb{C})$ satisfies*

$$m\big(\varphi(S) - \varphi(T)\big) = m(S - T), \qquad (S, T \in M_n(\mathbb{C})), \tag{3.3.7}$$

if and only if there are $U,\ V,\ R \in M_n(\mathbb{C})$, with U and V unitary matrices, such that

$$\varphi(T) = UT^\# V + R, \qquad (T \in M_n(\mathbb{C})),$$

where $T^\#$ stands for T, or T^{tr}, or T^*, or $\overline{T}$, the complex conjugation of T.

Proof. The "if part" is obvious. So, assume that (3.3.7) is satisfied and note that it follows from such an identity that $\varphi(S) - \varphi(T)$ is invertible if and only if $S - T$ is invertible. By Theorem 3.3.1, there are $A,\ B,\ R \in M_n(\mathbb{C})$ and an automorphism σ of $\mathbb{C}$ such that A and B are invertible matrices and either $\varphi(T) = AT_\sigma B + R$ or $\varphi(T) = A(T_\sigma)^{tr} B + R$ for all $T \in M_n(\mathbb{C})$. For any $z \in \mathbb{C}$, we see that

$$|z| = \mathrm{m}(z\mathbf{1}) = \mathrm{m}(\varphi(z\mathbf{1}) - \varphi(0)) = |\sigma(z)|\mathrm{m}(AB) = |\sigma(z)|.$$

Hence, σ is the identity or conjugation map and thus

$$\varphi(T) = AT^{\#}B + R, \qquad (T \in M_n(\mathbb{C})).$$

By (3.3.7), we have

$$\frac{1}{\|B^{-1}T^{\#^{-1}}A^{-1}\|} = \mathrm{m}(\varphi(T) - \varphi(0)) = \mathrm{m}(T - 0) = \frac{1}{\|T^{-1}\|} = \frac{1}{\|T^{\#^{-1}}\|}$$

for all invertible matrices $T \in M_n(\mathbb{C})$. Replacing T by $T^{\#^{-1}}$, we get that

$$\|B^{-1}TA^{-1}\| = \|T\|$$

for all invertible matrices $T \in M_n(\mathbb{C})$. By [13, Lemma 1.5], there are unitary matrices $U,\ V \in M_n(\mathbb{C})$ and scalars $\lambda,\ \mu \in \mathbb{C}$ such that $\lambda\mu = 1$, $A = \lambda U$ and $B = \mu V$. It follows that

$$\varphi(T) = AT^{\#}B + R = (\lambda U)T^{\#}(\mu V) + R = \lambda\mu UT^{\#}V + R = UT^{\#}V + R$$

for all $T \in M_n(\mathbb{C})$. □

The following is a variant of Theorem 3.3.2, with a similar proof. But, the only thing that should be observed is that Theorem 3.3.1 remains true when using sums in (3.1.1) instead of subtractions and $\varphi(0) = 0$ if (3.3.8) is satisfied. This is, in fact, an immediate consequence of Lemma 3.2.2.

Theorem 3.3.3. *A surjective map φ on $M_n(\mathbb{C})$ satisfies*

$$m\big(\varphi(S) + \varphi(T)\big) = m(S + T), \qquad (S, T \in M_n(\mathbb{C})), \tag{3.3.8}$$

if and only if there are unitary matrices $U,\ V \in M_n(\mathbb{C})$ such that

$$\varphi(T) = UT^{\#}V, \qquad (T \in M_n(\mathbb{C})).$$

Here again, $T^{\#}$ stands for T, or T^{tr}, or T^*, or $\overline{T}$, the complex conjugation of T.

3.4 Nonlinear maps preserving different moduli of operators

In this section, we describe maps φ from $\mathscr{B}(X)$ onto $\mathscr{B}(Y)$ satisfying (3.1.4) and (3.1.5) and establish some ingredients needed for the proof of the main results of this section. The obtained results generalize certain published results on linear or additive maps preserving the minimum and surjectivity moduli of operators; see for instance [13–15, 67, 77] and the references therein.

Theorem 3.4.1. *Assume that φ is a map from $\mathscr{B}(X)$ onto $\mathscr{B}(Y)$ satisfying (3.1.4). Then one of the following situations hold:*

(i) There are $R \in \mathscr{B}(Y)$ and bijective isometries $U : X \to Y$ and $V : Y \to X$, either both linear or both conjugate linear, such that

$$\varphi(T) = UTV + R, \qquad (T \in \mathscr{B}(X)). \tag{3.4.9}$$

(ii) There are bijective isometries $U : X^* \to Y$ and $V : Y \to X^*$, either both linear or both conjugate linear, such that

$$\varphi(T) = UT^*V + R, \qquad (T \in \mathscr{B}(X)). \tag{3.4.10}$$

Let us add a few comments to this theorem. First, the statement (ii) can not occur if any of X or Y is not reflexive. Second, let φ be a map from $\mathscr{B}(X)$ onto $\mathscr{B}(Y)$. Then if φ takes (3.4.9) then (3.1.4) is satisfied. However, if φ takes (3.4.10) then

$$\mathrm{m}(\varphi(T) - \varphi(S)) = \mathrm{m}(U(T-S)^*V) = \mathrm{m}((T-S)^*) = \mathrm{q}(T-S)$$

for all $S,\ T \in \mathscr{B}(X)$. Similarly, one has

$$\mathrm{q}(\varphi(T) - \varphi(S)) = \mathrm{q}(U(T-S)^*V) = \mathrm{q}((T-S)^*) = \mathrm{m}(T-S)$$

for all $S,\ T \in \mathscr{B}(X)$. From these, it is clear that

$$\mathrm{M}(\varphi(T) - \varphi(S)) = \mathrm{M}(T-S), \qquad (S,\ T \in \mathscr{B}(X)).$$

But, if there is a non-invertible surjective operator T_0 in $\mathscr{B}(X)$, then no map φ of the form (3.4.10) preserves the minimum or surjectivity moduli of the difference of operators. Indeed, assume that φ takes (3.4.10) and preserves the minimum modulus of the difference. Then, since $\mathrm{q}(T_0) > 0$ as T_0 is surjective, we have

$$\mathrm{m}(T_0) = \mathrm{m}(\varphi(T_0) - \varphi(0)) = \mathrm{q}(T_0) > 0.$$

Hence, T_0 is bounded from below as well, and thus it must be invertible. This is a contradiction. We finally point out that there exists a complex infinite-dimensional Banach space X such that every operator in $\mathscr{B}(X)$ is a scalar plus compact; see [4]. Thus for such a Banach space X, every surjective operator is invertible. On the other hand, if $\mathscr{H}$ is a complex infinite-dimensional Hilbert space, then there are non invertible surjective operators in $\mathscr{B}(\mathscr{H})$.

For our purposes, we establish a similar result to Havlicek and Šemrl [43] and Hou and Huang [48] by replacing the invertibility by surjectivity and boundedness below. Such a result plays a crucial role in the proof of the above theorem. Let $\Omega_{\mathrm{Inj}}(X)$, $\Omega_{\mathrm{Surj}}(X)$, $\Omega_{\mathrm{LB}}(X)$, $\Omega_{\text{Inj-or-Surj}}(X)$ and $\Omega_{\text{LB-or-Surj}}(X)$ be respectively the subsets of all injective operators, surjective operators, lower bounded operators, injective or surjective operators, and lower bounded or surjective operators of $\mathscr{B}(X)$. Assume that $\Omega(X)$ stands for any of these sets, and note that every operator T in $\Omega(X)$ is either injective or surjective and that $T \in \Omega(X)$ and

$$T \cdot \Omega(X) = \Omega(X) \cdot T = \Omega(X)$$

for all invertible operators $T \in \mathscr{B}(X)$.

Theorem 3.4.2. *If a surjective map φ from $\mathscr{B}(X)$ into $\mathscr{B}(Y)$ satisfies*

$$\varphi(S) - \varphi(T) \in \Omega(X) \iff S - T \in \Omega(X), \tag{3.4.11}$$

then φ is of the form (3.1.2) or (3.1.3).

In the proof of this theorem, we show that if φ satisfies (3.4.11), then φ is a bijective map preserving the invertibility of the difference of operators. But to establish this, we first characterize the adjacency of operators in term of operators in $\Omega(X)$, as shown in the following lemma, whose proof relies on some arguments similar to the ones given in [43, 46].

Lemma 3.4.3. *For two different operators S and T in $\mathscr{B}(X)$, the following statements are equivalent.*

(i) *S and T are adjacent; i.e., $S-T$ has rank one.*

(ii) *There is an operator $R\in\mathscr{B}(X)\backslash\{S,\ T\}$ such that either $N-S\in\Omega(X)$ or $N-T\in\Omega(X)$ for all $N\in\Omega(X)+\{R\}$.*

Proof. Note that none of the above conditions are effected if S and T are replaced by $S-F$ and $T-F$ for any $F\in\mathscr{B}(X)$. Thus, we may and shall prove this theorem when $S=0$.

We first establish the implication $(ii)\Rightarrow(i)$. Suppose that T has at least rank two and let us prove that for every $R\in\mathscr{B}(X)$ different from 0 and T, there is $N\in\mathscr{B}(X)$ such that $N-R\in\Omega(X)$ and both N and $N-T$ are not in $\Omega(X)$. So, let $R\in\mathscr{B}(X)$ be an operator different from 0 and T and assume that $Tz-Rz$ and Rx are linearly dependent for all $x,\ z\in X$. It follows, since $R\neq 0$, that both $T-R$ and R are rank one operators with the same image. Thus, T has at most rank one which contradicts the fact that T has at least rank two. Hence, there are nonzero vectors $x_1,\ x_2\in X$ such that Tx_2-Rx_2 and Rx_1 are linearly independent. If x_1 and x_2 are linearly dependent, then choose $u\in X$ linearly independent of x_1. Thus, $x_2+\lambda u$ and x_1 are linearly independent for all nonzero $\lambda\in\mathbb{C}$ and

$$T(x_2+\lambda u)-R(x_2+\lambda u)=Tx_2-Rx_2+\lambda(Tu-Ru)$$

and Rx_1 are linearly independent for small enough λ. Thus, we may and shall assume that there are linearly independent vectors $x_1,\ x_2\in X$ such that Tx_2-Rx_2 and Rx_1 are linearly independent.

Choose a scalar $\lambda\in\mathbb{C}$ such that $\lambda\mathbf{1}-T$ and $\lambda\mathbf{1}-R$ are both invertible, and set

$$x_3:=(\lambda\mathbf{1}-R)^{-1}(Tx_2-Rx_2)$$

and

$$x_4:=(\lambda\mathbf{1}-R)^{-1}(-Rx_1).$$

Put $X_0=\bigvee\{x_1,x_2,x_3,x_4\}$, and let us discuss several cases.

Case 1. Assume that $\dim(X_0)=4$, and let $f_1,\ f_2,\ f_3,\ f_4$ be linear functionals on X such that $\langle x_j,f_i\rangle=\delta_{ij}$ for all $1\leq i,\ j\leq 4$, where δ_{ij} is the Kronecker symbol. Set $X_1=\bigcap_{1\leq i\leq 4}\ker(f_i)$ so that $X=X_0\oplus X_1$, and let

$$\begin{aligned} N &= \lambda\mathbf{1}-\lambda x_1\otimes f_1+(Tx_2-\lambda x_2)\otimes f_2 \\ &+ (\lambda x_2-\lambda x_3-Rx_2+Rx_3)\otimes f_3 \\ &+ (\lambda x_1-\lambda x_4-Rx_1+Rx_4)\otimes f_4, \end{aligned}$$

and note that, since $Nx_1 = 0$, we see that N is not injective and 0 lies in $\sigma(N)$. As $\sigma(N)$ is finite and

$$\partial\sigma(N) \subset \sigma_{su}(N) \subset \sigma(N),$$

we see that $\sigma(N) = \sigma_{su}(N)$. Here, $\sigma_{su}(N)$ is the subjectivity spectrum of N and $\partial\sigma(N)$ is the boundary of $\sigma(N)$. It follows that the operator N is not surjective as well and thus $N \notin \Omega(X)$.

We also note that $(N-T)X_1 = (\lambda\mathbf{1}-T)X_1$ since N is nothing but $\lambda\mathbf{1}$ when restricted to X_1. As $\lambda\mathbf{1}-T$ is invertible, we have $\dim((\lambda\mathbf{1}-T)X_0) = 4$ and

$$X = (\lambda\mathbf{1}-T)X_0 \oplus (\lambda\mathbf{1}-T)X_1 = (\lambda\mathbf{1}-T)X_0 \oplus (N-T)X_1.$$

Thus $N-T$ can be surjective only if $(N-T)X_0 = (\lambda\mathbf{1}-T)X_0$. But this does not hold since $(N-T)x_2 = 0$ and thus

$$\dim((N-T)X_0) = \dim(\bigvee\{(N-T)x_1, (N-T)x_3, (N-T)x_4\}) \le 3.$$

It follows that $N-T$ is not surjective and is not injective. Thus, $N-T \notin \Omega(X)$.

Just as before, since $\lambda\mathbf{1}-R$ is invertible and $(N-R)X_1 = (\lambda\mathbf{1}-R)X_1$, we note that $N-R$ is invertible if and only if $(N-R)X_0 = (\lambda\mathbf{1}-R)X_0$. This always holds since

$$\begin{aligned}
(N-R)x_1 &= -Rx_1 = (\lambda\mathbf{1}-R)x_4,\\
(N-R)x_2 &= Tx_2 - Rx_2 = (\lambda\mathbf{1}-R)x_3,\\
(N-R)x_3 &= (\lambda\mathbf{1}-R)x_2,\\
(N-R)x_4 &= (\lambda\mathbf{1}-R)x_1,
\end{aligned}$$

and thus $N-R$ is invertible. It follows that $N-R \in \Omega(X)$.

Case 2. Assume that $\dim(X_0) = 3$, and $x_4 \in X_0 = \bigvee\{x_1, x_2, x_3\}$ so that

$$x_4 = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$$

for some scalars α_i, $1 \le i \le 3$. Let $f_1,\ f_2,\ f_3$ be linear functionals on X such that $\langle x_j, f_i\rangle = \delta_{ij}$ for all $1 \le i,\ j \le 3$, and note that, since x_3 and x_4 are linearly independent, either $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$. Assume that $\alpha_1 \neq 0$, and let

$$N = \lambda\mathbf{1} - \lambda x_1 \otimes f_1 + (Tx_2 - \lambda x_2) \otimes f_2 + (\lambda x_2 - \lambda x_3 - Rx_2 + Rx_3) \otimes f_3.$$

Just as in the previous case, one can show that both N and $N-T$ are neither injective nor surjective and thus both N and $N-T$ are not in $\Omega(X)$.

To show that $N-R \in \Omega(X)$, we only need to establish that $(N-R)X_0 = (\lambda\mathbf{1}-R)X_0$. Indeed, we have $(N-R)x_2 = (\lambda\mathbf{1}-R)x_3$, $(N-R)x_3 = (\lambda\mathbf{1}-R)x_2$ and

$$\begin{aligned}
(N-R)(x_1 - \alpha_3 x_2 - \alpha_2 x_3) &= -Rx_1 - \alpha_3(\lambda\mathbf{1}-R)x_3 - \alpha_2(\lambda\mathbf{1}-R)x_2\\
&= (\lambda\mathbf{1}-R)(x_4 - \alpha_3 x_3 - \alpha_2 x_2)\\
&= \alpha_1(\lambda\mathbf{1}-R)x_1.
\end{aligned}$$

From this, one infers that $(N-R)X_0 = (\lambda\mathbf{1}-R)X_0$ and thus $(N-R) \in \Omega(X)$.

Now, assume that $\alpha_1 = 0$, and set

$$N = \lambda\mathbf{1} - \lambda x_1 \otimes f_1 + (Tx_2 - \lambda x_2) \otimes f_2 + (\lambda x_1 - \lambda x_3 - Rx_1 + Rx_3) \otimes f_3.$$

Just as in the previously, one can show that both N and $N-T$ are not in $\Omega(X)$, and $(N-R)x_2 = (\lambda\mathbf{1}-R)x_3$, $(N-R)x_3 = (\lambda\mathbf{1}-R)x_1$ and

$$\begin{aligned}(N-R)(x_2-\alpha_3 x_1) &= (\lambda\mathbf{1}-R)x_3 - \alpha_3(\lambda\mathbf{1}-R)x_4\\ &= (\lambda\mathbf{1}-R)(x_4-\alpha_3 x_3)\\ &= \alpha_2(\lambda\mathbf{1}-R)x_2.\end{aligned}$$

From this, one infers that $(N-R)X_0 = (\lambda\mathbf{1}-R)X_0$ and $N-R \in \Omega(X)$.

Case 3. Assume that $\dim(X_0) = 3$, and $x_3 \in X_0 = \bigvee\{x_1, x_2, x_4\}$. This case is dealt with similar reasoning as in the previous case.

Case 4. Assume that $\dim(X_0) = 2$, and $x_3,\ x_4 \in X_0 = \bigvee\{x_1, x_2\}$ so that

$$\begin{aligned}x_3 &= \alpha_1 x_1 + \beta_1 x_2,\\ x_4 &= \alpha_2 x_1 + \beta_2 x_2,\end{aligned}$$

for some scalars α_i and β_i, $i = 1,\ 2$. Let f_1 and f_2 be linear functionals on X such that $\langle x_j, f_i\rangle = \delta_{ij}$ for all $i,\ j = 1,\ 2$, and set

$$N = \lambda\mathbf{1} - \lambda x_1 \otimes f_1 + (Tx_2 - \lambda x_2) \otimes f_2.$$

Just as previously, one can show that both N and $N-T$ are neither injective nor surjective and thus both N and $N-T$ are not in $\Omega(X)$.

To finish, note that

$$(N-R)x_2 = Tx_2 - Rx_2 = (\lambda\mathbf{1}-R)x_3 = \alpha_1(\lambda x_1 - Rx_1) + \beta_1(\lambda x_2 - Rx_2),$$

and

$$(N-R)x_1 = -Rx_1 = (\lambda\mathbf{1}-R)x_4 = \alpha_2(\lambda x_1 - Rx_1) + \beta_2(\lambda x_2 - Rx_2).$$

It follows that $(N-R)X_0 = (\lambda\mathbf{1}-R)X_0$ and $N-R \in \Omega(X)$ in this case too.

Conversely, let us establish the implication $(i) \Rightarrow (ii)$. Assume that $T := x \otimes f$ is a rank one operator, where $x \in X$ and $f \in X^*$. Let $R = 2T$ and let N be an operator in $\mathscr{B}(X)$ such that $N - R = N - 2T \in$

$\Omega(X)$, and let us show that $N \in \Omega(X)$ or $N - T \in \Omega(X)$. We shall discuss several cases.

Case 1. Assume that $\Omega(X)$ coincides with $\Omega_{\mathrm{LB}}(X)$, and let us first show that there is $u \in X$ such that $(N-2T)u = x$. If $N \notin \Omega_{\mathrm{LB}}(X)$, then there exists a unit vector sequence $\{x_n\}_n$ such that $\|Nx_n\| \to 0$. As $\{\langle x_n, f\rangle\}_n$ is a bounded sequence of complex numbers, we may assume, without loss of generality, that $\langle x_n, f\rangle \to a \in \mathbb{C}$. If $a = 0$, then

$$\|(N-R)x_n\| = \|(N-2T)x_n\| = \|Nx_n - 2\langle x_n, f\rangle x\| \to 0.$$

This contradicts the fact that $N - R = N - 2T$ is bounded below and shows that a is different from zero. As

$$\begin{aligned}\|(N-2T)x_n + 2ax\| &\leq \|(N-2T)x_n + 2Tx_n\| + \|2Tx_n - 2ax\|\\ &= \|Nx_n\| + \|2Tx_n - 2ax\| \to 0\end{aligned}$$

and the range of $N-2T$ is closed, we see that x lies in the range of $N-2T$ and thus there is u such that $(N-2T)u = x$; as claimed. It follows that

$$(N-2T-\lambda T) = (N-2T)(\mathbf{1} - \lambda u \otimes f) \tag{3.4.12}$$

for all $\lambda \in \mathbb{C}$. Note that, since $N-2T \in \Omega_{\mathrm{LB}}(X)$, we have

$$N-2T-\lambda T \in \Omega_{\mathrm{LB}}(X) \iff \mathbf{1} - \lambda u \otimes f \text{ is invertible}$$

for all $\lambda \in \mathbb{C}$. Since $N = N-2T-(-2)T \notin \Omega_{\mathrm{LB}}(X)$, we see that $\mathbf{1} - \lambda u \otimes f$ is not invertible when $\lambda = -2$ and thus $\sigma(u \otimes f) = \left\{0, -\frac{1}{2}\right\}$ since the spectrum of any rank one operator has at most one nonzero point. It follows that $N - T = N - 2T - (-1)T \in \Omega_{\mathrm{LB}}(X)$, and thus the implication $(i) \Rightarrow (ii)$ is established for $\Omega_{\mathrm{LB}}(X)$.

Case 2. If $\Omega(X)$ coincides with $\Omega_{\mathrm{Surj}}(X)$, then obviously there is $u \in X$ such that $(N-2T)u = x$ and thus (3.4.12) is satisfied. The rest of the proof of this case runs as in the proof of the previous case.

Case 3. If $\Omega(X)$ coincides with $\Omega_{\text{LB-or-Surj}}(X)$, then the conclusion is a consequence of case 1 and case 2 together.

Case 4. If $\Omega(X)$ coincides with $\Omega_{\mathrm{Inj}}(X)$ and then $N-R = N-2T$ is injective. Assume that N and $N-T$ are not injective. Thus, $Nu = 0$ and $(N-T)v = 0$ and $Nv = \langle v, f\rangle x$. If u and v are linearly dependent then $Nu = Nv = (N-T)v = 0$ implies that $Tv = 0$. Hence, $(N-2T)v = 0$ and this contradicts the fact that $N-2T$ is injective. Thus, u and v are linearly independent. On the other hand, we have

$$(N-2T)u = -2\langle u, f\rangle x \text{ and } (N-2T)v = (N-T)v - Tv = -\langle v, f\rangle x,$$

and

$$(N-2T)\big(\langle v,f\rangle u-2\langle u,f\rangle v\big)=0.$$

The injectivity of $N-2T$ implies that $\langle v,f\rangle u-2\langle u,f\rangle v=0$ and $\langle u,f\rangle=\langle v,f\rangle=0$. But, this is impossible.

Case 5. If $\Omega(X)$ coincides with $\Omega_{\text{Inj-or-Surj}}(X)$, then the conclusion is a consequence of case 2 and case 4. $\square$

The following lemma generalizes the corresponding one from [43, Lemma 3.2]. Its proof is on the straightforward side and uses exactly the same arguments given in [43]. We omit the details.

Lemma 3.4.4. *(Havlicek-Šemrl [43].) Let S, T be two operators in $\mathscr{B}(X)$ such that for every invertible operator N in $\mathscr{B}(X)$, we have*

$$S-N\in\Omega(X)\iff T-N\in\Omega(X).$$

Then $S=T$.

The following lemma is quoted from [70], and describes nonlinear maps that map finite rank operators onto themselves and preserve the rank oneness of the difference of operators. Recall that a map A from X into Y is called *semilinear* if it is additive and there is an automorphism σ of $\mathbb{C}$ such that $A(\lambda x)=\sigma(\lambda)Ax$ for all $x\in X$ and $\lambda\in\mathbb{C}$. Such a map is sometimes called σ-semilinear when the automorphism σ is specified.

Lemma 3.4.5. *(Petek-Šemrl [70].) Assume that X and Y are Banach spaces of dimensions at least 2, and let φ be a bijective map from $\mathscr{F}(X)$ into $\mathscr{F}(Y)$ such that whenever S, T are operators in $\mathscr{F}(X)$ then one has*

$$S-T \text{ has rank one} \iff \varphi(S)-\varphi(T) \text{ has rank one.}$$

Then one of the following situations hold:

(i) There are an automorphism σ of $\mathbb{C}$, $R\in\mathscr{B}(Y)$, and bijective σ-semilinear maps $A:X\to Y$ and $B:X^*\to Y^*$ such that $T\mapsto\varphi(T)-R$ is an additive map defined by

$$\varphi(x\otimes f)-R=Ax\otimes Bf,\qquad (x\in X,\ f\in X^*).$$

(ii) There are an automorphism σ of $\mathbb{C}$, $R\in\mathscr{B}(Y)$, and bijective σ-semilinear maps $A:X\to Y^*$ and $B:X^*\to Y$ such that $T\mapsto\varphi(T)-R$ is an additive map defined by

$$\varphi(x\otimes f)-R=Bf\otimes Ax,\qquad (x\in X,\ f\in X^*).$$

Now, we are ready to prove Theorem 3.4.2 whose proof breaks down into four steps

Proof of Theorem 3.4.2. Assume that φ satisfies (3.4.11) and note that $\varphi - \varphi(0)$ satisfies (3.4.11) as well. Thus, after replacing φ by $\varphi - \varphi(0)$, we may and shall assume that $\varphi(0) = 0$.

Step 1. φ is bijective.

Assume that $\varphi(S_1) = \varphi(S_2)$ for some operators S_1 and S_2 in $\mathscr{B}(X)$, and let us show that $S_1 = S_2$. For every operator $N \in \mathscr{B}(X)$, we have

$$\begin{aligned} S_1 - N \in \Omega(X) \quad &\Longleftrightarrow \quad \varphi(S_1) - \varphi(N) \in \Omega(X) \\ &\Longleftrightarrow \quad \varphi(S_2) - \varphi(N) \in \Omega(X) \\ &\Longleftrightarrow \quad S_2 - N \in \Omega(X). \end{aligned}$$

By Lemma 3.4.4, we see that $S_1 = S_2$ and thus φ is injective. It is in fact bijective since it is supposed to be surjective.

Step 2. Either there exist bijective semilinear maps $A : X \to Y$ and $B : X^* \to Y^*$ with the same associated automorphism of $\mathbb{C}$ such that

$$\varphi(x \otimes f) = (Ax) \otimes (Bf), \ (x \in X, \ f \in X^*), \tag{3.4.13}$$

or there exist bijective semilinear maps $A : X \to Y^*$ and $B : X^* \to Y$ with the same associated automorphism of $\mathbb{C}$ such that

$$\varphi(x \otimes f) = (Bf) \otimes (Ax), \ (x \in X, \ f \in X^*). \tag{3.4.14}$$

By Lemma 3.4.3, φ preserves adjacency in both directions. Every rank one operator is adjacent to zero, every rank two operator is adjacent to a rank one operator, etc. Consequently, φ maps the ideal of all finite rank operators $\mathscr{F}(X)$ onto itself. Now, Lemma 3.4.5 ensures that φ takes one of the above forms.

In the rest of the proof, we may and shall assume that φ takes the first form (3.4.13) since the case when φ has the second form (3.4.14) is dealt in a similar way.

Step 3. φ preserves the invertibility in both directions.

Let us first show that $\varphi(\mathbf{1})$ is invertible and let us discuss several cases.

Case 1. Assume that $\Omega(X) = \Omega_{\text{Inj}}(X)$, and note that $\varphi(\mathbf{1})$ is injective. Let us show that $\varphi(\mathbf{1})$ is surjective. For every nonzero vector $x \in X$, let $f \in X^*$ such that $\langle x, f\rangle = 1$, and note that $\mathbf{1} - x \otimes f$ is not injective and so is $\varphi(\mathbf{1}) - (Ax) \otimes (Bf)$. Thus $\big(\varphi(\mathbf{1}) - (Ax) \otimes (Bf)\big)\, y = 0$ for some nonzero $y \in X$. This shows that $\varphi(\mathbf{1})\, y = \langle y, Bf\rangle Ax$. As $\varphi(\mathbf{1})$ is injective, we see that $\langle y, Bf\rangle \neq 0$ and Ax lies in the range of $\varphi(\mathbf{1})$. Since A is semilinear and bijective, we infer that Ax lies in the range of $\varphi(\mathbf{1})$ for all $x \in X$ and $\varphi(\mathbf{1})$ is surjective; as desired.

Case 2. Assume that $\Omega(X) = \Omega_{\text{Surj}}(X)$, and note that $\varphi(\mathbf{1})$ is surjective and thus $\varphi(\mathbf{1})^*$ is bounded below. Let $f \in X^*$ be a nonzero arbitrary linear functional, and let $x \in X$ such that $\langle x, f\rangle = 1$. We have $\mathbf{1} - x \otimes f$ is not surjective, and so is $\varphi(\mathbf{1}) - (Ax) \otimes (Bf)$ and thus $\varphi(\mathbf{1})^* - (Bf) \otimes (Ax)$ is not bounded below. Since $\varphi(\mathbf{1})^*$ has a closed range, we see that $\varphi(\mathbf{1})^* - (Bf) \otimes (Ax)$ has a closed range too and thus $\varphi(\mathbf{1})^* - (Bf) \otimes (Ax)$ is not injective. Hence, there is $g \in Y^*$ such that $\big(\varphi(\mathbf{1})^* - (Bf) \otimes (Ax)\big)\, g = 0$ and thus $\varphi(\mathbf{1})^* g = \langle g, Ax\rangle Bf$. As $\varphi(\mathbf{1})^*$ is injective, we see that $\langle g, Ax\rangle \neq 0$ and Bf lies in the range of $\varphi(\mathbf{1})^*$. Since B is semilinear and bijective, we infer that Bf lies in the range of $\varphi(\mathbf{1})^*$ for all $f \in X^*$ and $\varphi(\mathbf{1})^*$ is surjective. It follows that $\varphi(\mathbf{1})$ is bounded below but since it is surjective, we in fact see that $\varphi(\mathbf{1})$ is invertible.

Case 3. Assume that $\Omega(X) = \Omega_{\text{Inj-or-Surj}}(X)$, and note that $\varphi(\mathbf{1})$ is either injective or surjective. If $\varphi(\mathbf{1})$ is injective, let $x \in X$ be a nonzero vector in X and $f \in X^*$ such that $\langle x, f\rangle = 1$. Note that $\mathbf{1} - x \otimes f$ is not injective and is not surjective, and thus so is $\varphi(\mathbf{1}) - (Ax) \otimes (Bf)$. If follows that $\big(\varphi(\mathbf{1}) - (Ax) \otimes (Bf)\big)\, y = 0$ for some nonzero $y \in X$, and thus just as Case 1 one sees that $\varphi(\mathbf{1})$ is invertible.

If $\varphi(\mathbf{1})$ is surjective, then $\varphi(\mathbf{1})^*$ is bounded below. Let $f \in X^*$ be a nonzero arbitrary linear functional, and let $x \in X$ such that $\langle x, f\rangle = 1$. Clearly, $\mathbf{1} - x \otimes f$ is not injective and is not surjective, and so is $\varphi(\mathbf{1}) - (Ax) \otimes (Bf)$. Hence, there is $g \in Y^*$ such that $\big(\varphi(\mathbf{1})^* - (Bf) \otimes (Ax)\big)\, g = 0$ and thus $\varphi(\mathbf{1})^* g = \langle g, Ax\rangle Bf$. As $\varphi(\mathbf{1})^*$ is injective, we see that $\langle g, Ax\rangle \neq 0$ and Bf lies in the range of $\varphi(\mathbf{1})^*$. Since B is semilinear and bijective, we infer that Bf lies in the range of $\varphi(\mathbf{1})^*$ for all $f \in X^*$ and $\varphi(\mathbf{1})^*$ is surjective. It follows that $\varphi(\mathbf{1})$ is bounded below but since it is surjective, we in fact see that $\varphi(\mathbf{1})$ is invertible.

Case 4. Assume that $\Omega(X) = \Omega_{\text{LB}}(X)$, and note that $\varphi(\mathbf{1})$ is bounded below. Let $x \in X$ be a nonzero vector in X and $f \in X^*$ such that $\langle x, f\rangle = 1$. Note that $\mathbf{1} - x \otimes f$ is not bounded below, and thus so is $\varphi(\mathbf{1}) - (Ax) \otimes (Bf)$. It follows that $\big(\varphi(\mathbf{1}) - (Ax) \otimes (Bf)\big)\, y = 0$ for some nonzero $y \in X$, and thus just as Case 1 one sees that $\varphi(\mathbf{1})$ is invertible.

Case 5. Assume that $\Omega(X) = \Omega_{\text{LB-or-Surj}}(X)$, and note that $\varphi(\mathbf{1})$ is either bounded below or surjective. If $\varphi(\mathbf{1})$ is surjective, then $\varphi(\mathbf{1})^*$ is bounded below. Let $f \in X^*$ be a nonzero arbitrary linear functional, and let $x \in X$ such that $\langle x, f\rangle = 1$. Clearly, $\mathbf{1} - x \otimes f$ is not bounded below and is not surjective, and so is $\varphi(\mathbf{1}) - (Ax) \otimes (Bf)$. Hence, there is $g \in Y^*$ such that $\big(\varphi(\mathbf{1})^* - (Bf) \otimes (Ax)\big) g = 0$ and thus $\varphi(\mathbf{1})^* g = \langle g, Ax\rangle Bf$. As $\varphi(\mathbf{1})^*$ is injective, we see that $\langle g, Ax\rangle \neq 0$ and Bf lies in the range of $\varphi(\mathbf{1})^*$. Since B is semilinear and bijective, we infer that Bf lies in the range of $\varphi(\mathbf{1})^*$ for all $f \in X^*$ and $\varphi(\mathbf{1})^*$ is surjective. It follows that $\varphi(\mathbf{1})$ is bounded below but since it is surjective, we in fact see that $\varphi(\mathbf{1})$ is invertible.

Now, we show that $\varphi(T_0)$ is invertible for all invertible operator $T_0 \in \mathscr{B}(X)$. Consider

$$\phi(T) := \varphi(T_0 T) \quad (T \in \mathscr{B}(X)),$$

and note that ϕ is a bijective map on $\mathscr{B}(X)$ satisfying (3.4.11). By what has shown above, we see that $\phi(\mathbf{1}) = \varphi(T_0)$ is invertible. Since φ^{-1} satisfies (3.4.11) as well, we deduce from what has been discussed that φ preserves the invertibility in both directions.

Step 4. φ satisfies (3.1.1), and takes the desired forms.

Fix an arbitrary operator $T \in \mathscr{B}(X)$ and define

$$\psi(S) := \varphi(S + T) - \varphi(T)$$

for all $S \in \mathscr{B}(X)$. Clearly, ψ is a surjective map satisfying (3.4.11) and thus, by the preceding step, ψ preserves the invertibility in both directions. In particular, for every $S \in \mathscr{B}(X)$, we have $S - T$ is invertible if and only if $\psi(S-T) = \varphi(S) - \varphi(T)$ is invertible. So, φ satisfies (3.1.1) and Theorem 3.1.1 shows that φ takes the desired forms. □

All the ingredients are collected and we therefore are in a position to prove Theorem 3.4.1.

Proof of Theorem 3.4.1. Assume that φ is a map from $\mathscr{B}(X)$ onto $\mathscr{B}(Y)$ satisfying

$$\mathrm{c}(\varphi(S) - \varphi(T)) = \mathrm{c}(S - T) \tag{3.4.15}$$

for all $S,\ T \in \mathscr{B}(X)$, and note that φ is a bijective map satisfying (3.4.11). By Theorem 3.4.2, there exists an operator $R \in \mathscr{B}(Y)$ and either there are bijective continuous mappings $A : X \to Y$ and $B : Y \to X$ both linear or both conjugate linear such that

$$\varphi(T) = ATB + R, \qquad T \in \mathscr{B}(X), \tag{3.4.16}$$

or there are bijective continuous mappings $A : X^* \to Y$ and $B : Y \to X^*$ both linear or both conjugate linear such that

$$\varphi(T) = AT^*B + R, \qquad T \in \mathscr{B}(X). \tag{3.4.17}$$

Assume without loss of generality that the first possibility holds, and note that

$$\frac{1}{\|T\|} = \mathrm{c}\left(T^{-1}\right) = \mathrm{c}\left(\varphi(T^{-1}) - \varphi(0)\right) = \mathrm{c}\left(AT^{-1}B\right) = \frac{1}{\|B^{-1}TA^{-1}\|}$$

for all invertible operators $T \in \mathscr{B}(X)$. By Lemma 3.2.3, there are isometries $U : X \to Y$ and $V : Y \to X$ both linear or both conjugate linear, and scalars λ and μ such that $A = \lambda U$ and $B = \mu V$ and $\lambda\mu = 1$. Thus

$$\varphi(T) = ATB + R = (\lambda U)T(\mu V) + R = UTV + R$$

for all $T \in \mathscr{B}(X)$; as desired. □

3.5 Concluding remarks

With no extra effort, the proof of Theorem 3.4.2 establishes the following result.

Theorem 3.5.1. *If a surjective map φ from $\mathscr{B}(X)$ into $\mathscr{B}(Y)$ satisfies*

$$\varphi(S) + \varphi(T) \in \Omega(X) \iff S + T \in \Omega(X), \tag{3.5.18}$$

then φ takes (3.1.2) or (3.1.3) with $R = 0$.

This theorem allows one to describe surjective maps preserving the minimum, surjectivity and maximum moduli of the sum of operators and obtain the following variant of Theorem 3.4.1.

Theorem 3.5.2. *Assume that φ is a map from $\mathscr{B}(X)$ onto $\mathscr{B}(Y)$ satisfying*

$$c(\varphi(S) + \varphi(T)) = c(S + T)$$

for all $S,\ T \in \mathscr{B}(X)$. Then φ takes either (3.4.9) or (3.4.10) with $R = 0$.

With the obtained results and techniques of this paper, one could obtain the complete description of maps from $\mathscr{B}(X)$ onto $\mathscr{B}(Y)$ preserving different spectral sets and quantities (such as the ascent, descent, etc) of the difference and sum of operators. We finally let $\Omega_{\mathrm{Linv}}(X)$, $\Omega_{\mathrm{Rinv}}(X)$ and $\Omega_{\mathrm{LRinv}}(X)$ be the sets of all left invertible operators, right invertible operators and left or right invertible operators, respectively, of $\mathscr{B}(X)$, and point out that similar results to Theorem 3.4.2 and Theorem 3.5.1 can be obtained for these sets.

Chapter 4

Nonlinear maps preserving the reduced minimum modulus of operators

Résumé

Soit X et Y des espaces de Banach complexes de dimension infine, et soit $\mathscr{B}(X)$ (resp. $\mathscr{B}(Y)$) l'algèbre de tous les opérateurs linéaires et bornés sur X (resp. sur Y). Nous décrivons des applications bicontinues et bijectives φ de $\mathscr{B}(X)$ sur $\mathscr{B}(Y)$, qui satisfont

$$\gamma(\varphi(S \pm \varphi(T)) = \gamma(S \pm T)$$

pour tous $S,\ T \in \mathscr{B}(X)$, où $\gamma(T)$ est la conorme d'un opérator T. Un résultat analogue pour le cas de dimension finie est obtenu.

Abstract

Let X and Y be infinite-dimensional complex Banach spaces, and let $\mathscr{B}(X)$ (resp. $\mathscr{B}(Y)$) denote the algebra of all bounded linear operators on X (resp. on Y). We describe bijective bicontinuous maps φ from $\mathscr{B}(X)$ to $\mathscr{B}(Y)$ satisfying

$$\gamma(\varphi(S \pm \varphi(T)) = \gamma(S \pm T)$$

for all $S,\ T \in \mathscr{B}(X)$, where $\gamma(T)$ is the reduced minimum modulus of an operator T. An analogue result for the finite-dimensional case is obtained.

4.1 Introduction

Numerous studies have been done on the subject of nonlinear preserver problems. These problems, in the most general setting, demand the characterization of maps between algebras that leave a certain property, a particular relation, or even a subset invariant without assuming in advance algebraic conditions such as linearity, additivity or multiplicity; see for instance [27, 31, 35, 40–43, 45, 47–50, 53, 58, 60, 61, 68, 69, 71, 72, 80, 84]. The characterization of surjective maps on the algebra $M_n(\mathbb{C})$ of all complex $n \times n$-matrices preserving the spectral radius of the difference of matrices was given in [11] by Bhatia, Šemrl and Sourour. In [69], Molnár studied maps preserving the spectrum of matrix or Hilbert space operator products. His results have been extended in many directions for uniform algebras and semisimple commutative Banach algebras, and a number of results is obtained on maps preserving several spectral and local spectral quantities of operator or matrix product, or Jordan product, or Jordan triple product, or difference; see for instance [16–19, 27–30, 35, 40–43, 47, 49, 50, 53, 58, 60, 61, 68, 71, 72, 80] and the references therein.

Recently, Bourhim, Mashreghi and Stepanyan described in [20] nonlinear maps preserving the minimum and surjectivity moduli of the difference of operators and matrices, and thus extending the main results of several papers to the nonlinear setting; see for instance [13] and the references therein. However, the corresponding problem of characterizing nonlinear maps preserving the reduced minimum modulus was naturally left therein [20]. It is the aim of this note to describe such maps and show that a bijective bicontinuous map on the algebra of all bounded linear operators on a complex Banach space preserves the reduced minimum modulus of the differences of operators if and only if it is an isometry translated by an operator. The proof of such a promised result uses some arguments that are influenced by ideas from several papers including [13, 20].

4.2 Preliminaries

Let $M_n(\mathbb{C})$ denote, as usual, the algebra of all $n \times n$ complex matrices, and let T^{tr} denote the transpose of any matrix $T \in M_n(\mathbb{C})$. Let $\mathscr{B}(X)$ (resp. $\mathscr{B}(Y)$) be the algebra of all bounded linear operators on a complex Banach space X (resp. Y). The dual space of X is denoted by X^*, and the Banach space adjoint of an operator $T \in \mathscr{B}(X)$ is denoted by T^*. The *minimum modulus* of an operator $T \in \mathscr{B}(X)$ is $\mathrm{m}(T) := \inf\{\|Tx\| : x \in X,\ \|x\| = 1\}$, and is positive precisely when T is bounded below; i.e., T is injective and has a closed range. The *surjectivity modulus* of T is $\mathrm{q}(T) := \sup\{\varepsilon \geq 0 : \varepsilon B_X \subseteq T(B_X)\}$, and is positive if and only if T is surjective. Here, B_X is the closed unit ball of X. While, the *maximum modulus* of T is defined by $\mathrm{M}(T) := \max\{\mathrm{m}(T), \mathrm{q}(T)\}$, and is positive precisely when either T is bounded below or T is surjective. Note that $\mathrm{m}(T^*) = \mathrm{q}(T)$ and $\mathrm{q}(T^*) = \mathrm{m}(T)$ for all $T \in \mathscr{B}(X)$, and consequently $\mathrm{M}(T^*) = M(T)$ for all $T \in \mathscr{B}(X)$. Finally, recall that the reduced minimum modulus of T is defined by

$$\gamma(T) := \begin{cases} \inf\{\|Tx\| : \operatorname{dist}(x, Ker\,T) \geq 1\} & \text{if } T \neq 0, \\ \infty & \text{if } T = 0 \end{cases}$$

and is positive if and only if the range of T is closed. It is easy to see that $\gamma(T^*) = \gamma(T)$ and $\gamma(T) \geq \mathrm{M}(T)$. Moreover, we note that $\gamma(T) = \mathrm{M}(T)$ if $\mathrm{M}(T) > 0$, and

$$\gamma(T) = \mathrm{M}(T) = \mathrm{m}(T) = \mathrm{q}(T) = \|T^{-1}\|^{-1}$$

whenever T is invertible.

In [20], the description of surjective maps φ from $\mathscr{B}(X)$ to $\mathscr{B}(Y)$ satisfying

$$\mathrm{c}(\varphi(S)-\varphi(T)) = \mathrm{c}(S-T),\ (S,\ T \in \mathscr{B}(X)),$$

is obtained. Here, $\mathrm{c}(\cdot)$ stands either for the minimum modulus, or the surjectivity modulus, or the maximum modulus. But naturally one may ask if similar description can be established for the reduced minimum modulus instead of the spectral quantity $\mathrm{c}(\cdot)$. So, we have the following problem.

Problem 4.2.1. *Which surjective maps $\varphi : \mathscr{B}(X) \to \mathscr{B}(Y)$ satisfy*

$$\gamma(\varphi(S)-\varphi(T)) = \gamma(S-T) \tag{4.2.1}$$

for all $S,\ T \in \mathscr{B}(X)$?

Note that any map φ on $\mathscr{B}(X)$ satisfying (4.2.1) is injective. Indeed, if $\varphi(S) = \varphi(T)$, then

$$\infty = \gamma(0) = \gamma(\varphi(S)-\varphi(T)) = \gamma(S-T),$$

and thus $S-T=0$. Hence, φ is injective; as claimed. In view of this observation, we shall assume that our map φ is always bijective.

The aim of this note is to answer this problem but under the extra condition that φ is bicontinuous. To do so, we shall assume in the sequel that X and Y are infinite dimensional complex Banach spaces, and state two results needed for the proof of our main result.

4.3 Main result and its proof

We manage to describe the operators φ that satisfy the condition (4.2.1) and are bicontinuous. We tried to avoid the last condition, but we have not found the way still. The following theorem is the main result of this note.

Theorem 4.3.1. *A bicontinuous bijective map φ from $\mathscr{B}(X)$ into $\mathscr{B}(Y)$ satisfies (4.2.1) if and only if one of the following situations hold:*

(i) There are $R \in \mathscr{B}(Y)$ and bijective isometries $U : X \to Y$ and $V : Y \to X$, either both linear or both conjugate linear, such that

$$\varphi(T) = UTV + R, \qquad (T \in \mathscr{B}(X)). \tag{4.3.2}$$

(ii) There are $R \in \mathscr{B}(Y)$ and bijective isometries $U : X^* \to Y$ and $V : Y \to X^*$, either both linear or both conjugate linear, such that

$$\varphi(T) = UT^*V + R, \qquad (T \in \mathscr{B}(X)). \tag{4.3.3}$$

The second statement can not occur if any of X or Y is not reflexive.

Proof. Note that, since the "if" part obviously holds, we only need to establish the "only if" part. Assume that φ is a bicontinuous bijective map from $\mathscr{B}(X)$ into $\mathscr{B}(Y)$ satisfying (4.2.1). Note that we may and shall assume that $\varphi(0)=0$ since $\varphi-\varphi(0)$ satisfies (4.2.1) as well.

We first show that

$$\mathrm{M}(\varphi(T))=0 \Longleftrightarrow \mathrm{M}(T)=0, \qquad (T\in\mathscr{B}(X)).$$

Indeed, assume that $T_0\in\mathscr{B}(X)$ is an operator for which $\mathrm{M}(T_0)>0$ and let us show that $\mathrm{M}(R_0)>0$ where $R_0:=\varphi(T_0)$. Since $\mathrm{M}(T_0)>0$, then $\gamma(T_0)=\mathrm{M}(T_0)>0$ and consequently $\gamma(R_0)=\gamma(\varphi(T_0)-\varphi(0))=\gamma(T_0-0)>0$, so the operator R_0 has a closed range. To see that $\mathrm{M}(R_0)>0$ it suffices to show that R_0 is surjective or $\ker(R_0)$ is trivial. Assume by the way of contradiction that R_0 is not surjective and $\ker(R_0)$ is not trivial, and pick up two unit vectors $x\notin \mathrm{range}(R_0)$ and $y\in\ker(R_0)$. Let $f\in X^*$ be a linear functional such that $\langle y,f\rangle=1$, and $r>0$ be a positive number. Since $x\notin\mathrm{range}(R_0)$, we have $\ker(R_0-rx\otimes f)=\ker(R_0)\bigcap\ker(f)$ and

$$\begin{aligned} r &= ||(R_0-rx\otimes f)y|| \\ &\geq \gamma(R_0-rx\otimes f)\mathrm{dist}(y,\ker(R_0-rx\otimes f)) \\ &\geq \delta\gamma(R_0-rx\otimes f), \end{aligned}$$

where $\delta:=\mathrm{dist}(y,\ker(f))$ which is of course positive. As φ is bijective, we can write

$$r\geq\delta\gamma(R_0-rx\otimes f)=\delta\gamma(\varphi(T_0)-\varphi\varphi^{-1}(rx\otimes f)=\delta\gamma(T_0-\varphi^{-1}(rx\otimes f)).$$

As φ is supposed to be bicontinuous at 0, which means particularly that it's inverse is continuous at 0, so $S_r=\varphi^{-1}(rx\otimes f)\to 0$, when $r\to 0$. As the set of all operators with positive maximum modulus is open, it follows that $\mathrm{M}(T_0-S_r)>0$ and thus

$$r\geq\delta\gamma(T_0-S_r)=\delta\mathrm{M}(T_0-S_r)$$

for all sufficiently small numbers r. As the maximum modulus is a continuous function, the right side of the inequality tends to $\delta\mathrm{M}(T_0)>0$ as r goes to 0, and thus one gets a contradiction. We therefore have $\mathrm{M}(R_0)=\mathrm{M}(\varphi(T_0))>0$.

Noticing that φ^{-1} satisfies the same conditions as φ, we conclude that φ preserves the zeros of $\mathrm{M}(.)$ in the other direction also.

Second, let us show that $\mathrm{M}(S-T)=0$ if and only if $\mathrm{M}(\varphi(S)-\varphi(T))=0$ for all $S,\ T\in\mathscr{B}(X)$. Fix an operator $T\in\mathscr{B}(X)$, and define $\psi(S):=\varphi(S+T)-\varphi(T)$ for all $S\in\mathscr{B}(X)$. It is easy to see that ψ is a bicontinuous bijective map from $\mathscr{B}(X)$ into $\mathscr{B}(Y)$ satisfying (4.2.1) and for which $\psi(0)=0$. By what has shown above, we see that

$$\mathrm{M}(\psi(S))>0 \iff \mathrm{M}(S)>0$$

for all $S\in\mathscr{B}(X)$. Replacing S by $S-T$, we see that

$$\mathrm{M}(\varphi(S)-\varphi(T))=\mathrm{M}(\psi(S-T))>0 \iff \mathrm{M}(S-T)>0$$

for all $S,\ T \in \mathscr{B}(X)$.

Since $\gamma(T) = \mathrm{M}(T)$ for all $T \in \mathscr{B}(X)$ for which $\mathrm{M}(T) > 0$, one concludes from the last step and (4.2.1) that

$$\mathrm{M}(\varphi(S) - \varphi(T)) = \mathrm{M}(S - T)$$

for all $S,\ T \in \mathscr{B}(X)$. Therefore, the desired conclusion immediately holds by applying [20, Theorem 4.1]. □

We also obtain an analogue result for the finite-dimensional case.

Theorem 4.3.2. *A bicontinuous bijective map φ on $M_n(\mathbb{C})$ satisfies (4.2.1) if and only if there are $U,\ V,\ R \in M_n(\mathbb{C})$, with U and V unitary matrices, such that*

$$\varphi(T) = UT^{\#}V + R, \qquad (T \in M_n(\mathbb{C})), \tag{4.3.4}$$

where $T^{\#}$ stands for T, or T^{tr}, or T^*, or $\overline{T}$, the complex conjugation of T.

Proof. Assume that φ is a bicontinuous bijective map on $M_n(\mathbb{C})$ satisfying (4.2.1). Just as for the proof of the Theorem 4.3.1, one shows that

$$\mathrm{M}(\varphi(S) - \varphi(T)) = \mathrm{M}(S - T), \qquad (S,\ T \in M_n(\mathbb{C})),$$

and consequently, using [20, Theorem 3.2], we obtain the result. □

One may think about describing maps φ from $\mathscr{B}(X)$ onto $\mathscr{B}(Y)$ satisfying

$$\gamma(\varphi(S) + \varphi(T)) = \gamma(S + T) \tag{4.3.5}$$

for all $S,\ T \in \mathscr{B}(X)$. But for such a map φ, it is easy to see that $\varphi(0) = 0$ and (4.2.1) is automatically satisfied. Indeed, take $T = S = 0$ and note that (4.3.5) implies that $\infty = \gamma(0+0) = \gamma(2\varphi(0))$ and thus $\varphi(0) = 0$. On the other hand, fix an operator $T \in \mathscr{B}(X)$ and take $S = -T$, and note that (4.3.5) implies that $\gamma(\varphi(-T) + \varphi(T)) = \gamma(-T + T) = \infty$, and thus $\varphi(-T) = -\varphi(T)$. This shows that φ automatically satisfies (4.2.1).

These observations tell us that Theorem 4.3.1 and Theorem 4.3.2 remain valid when (4.2.1) is replaced by (4.3.5), but in their conclusions R must be 0.

Chapter 5

Nonlinear maps between Banach algebras preserving the spectrum

Résumé

Soit $\mathscr{A}$ et $\mathscr{B}$ des algèbres de Banach complexes unitaires semisimples et soit φ_1 et φ_2 des applications surjectives de $\mathscr{A}$ sur $\mathscr{B}$. Nous montrons que si le socle de $\mathscr{A}$ est un idéal essentiel de $\mathscr{A}$, et φ_1 et φ_2 satisfont

$$\sigma\big(\varphi_1(a)\varphi_2(b)\big) = \sigma(ab)$$

pour tous $a, b \in \mathscr{A}$, alors $\varphi_1\varphi_2(\mathbf{1})$ et $\varphi_1(\mathbf{1})\varphi_2$ coïncident et sont des isomorphismes de Jordan. Nous montrons aussi qu'une application surjective φ de $\mathscr{A}$ sur $\mathscr{B}$ satisfait

$$\sigma\big(\varphi(a)\varphi(b)\varphi(a)\big) = \sigma(aba)$$

pour tous $a, b \in \mathscr{A}$ si et seulement si $\varphi(\mathbf{1})$ est un élément inversible centrale de $\mathscr{B}$ pour lequel $\varphi(\mathbf{1})^3 = \mathbf{1}$ et $\varphi(\mathbf{1})^2\varphi$ est un isomorphisme de Jordan.

Abstract

Let $\mathscr{A}$ and $\mathscr{B}$ be unital semisimple complex Banach algebras, and let φ_1 and φ_2 be maps from $\mathscr{A}$ onto $\mathscr{B}$. We show that if the socle of $\mathscr{A}$ is an essential ideal of $\mathscr{A}$, and φ_1 and φ_2 satisfy

$$\sigma\big(\varphi_1(a)\varphi_2(b)\big) = \sigma(ab)$$

for all $a, b \in \mathscr{A}$, then $\varphi_1\varphi_2(\mathbf{1})$ and $\varphi_1(\mathbf{1})\varphi_2$ coincide and are Jordan isomorphisms. We also show that a map φ from $\mathscr{A}$ onto $\mathscr{B}$ satisfies

$$\sigma\big(\varphi(a)\varphi(b)\varphi(a)\big) = \sigma(aba)$$

for all $a, b \in \mathscr{A}$ if and only if $\varphi(\mathbf{1})$ is a central invertible element of $\mathscr{B}$ for which $\varphi(\mathbf{1})^3 = \mathbf{1}$ and $\varphi(\mathbf{1})^2\varphi$ is a Jordan isomorphism.

5.1 Introduction

Throughout this paper, let $\mathscr{A}$ and $\mathscr{B}$ be unital complex Banach algebras. For an element a in $\mathscr{A}$, let $\sigma(a)$ denote the spectrum of a. It is a nonempty compact set and its maximum modulus coincides with the spectral radius of a defined by $\mathrm{r}(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$. In [3, 24], the question whether for given two elements $a,\ b \in \mathscr{A}$ the condition $\sigma(ax) = \sigma(bx)$ for all $x \in \mathscr{A}$ entails $a = b$ was studied, and an affirmative answer has been obtained in [3, 24] for various special cases and for some classes of algebras, including C^*-algebras. Among other results and problems, they also considered the problem of describing maps φ from $\mathscr{A}$ onto $\mathscr{B}$ satisfying

$$\sigma\big(\varphi(a)\varphi(b)\big) = \sigma(ab), \qquad (a, b \in \mathscr{A}). \tag{5.1.1}$$

This problem was initiated by Molnár [69] when $\mathscr{A}$ and $\mathscr{B}$ coincide with the algebra $\mathscr{B}(\mathscr{H})$ of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ and his result has been extended in different directions for some special algebras; see [1, 27, 47, 59, 81] and the references therein. In [47], Hou and Di described, in particular, maps preserving the numerical ranges of products of Hilbert space operators. In [59], maps preserving the nilpotency of operator products are characterized.

In this paper, we solve the problem (5.1.1) in general scope. We show, in particular, that if the socle of $\mathscr{A}$ is an essential ideal of $\mathscr{A}$, then such a map φ is a Jordan isomorphism multiplied by a central invertible element of $\mathscr{B}$. We also describe maps φ from $\mathscr{A}$ onto $\mathscr{B}$ satisfying

$$\sigma\big(\varphi(a)\varphi(b)\varphi(a)\big) = \sigma(aba), \qquad (a, b \in \mathscr{A}) \tag{5.1.2}$$

when the socle of $\mathscr{A}$ is an essential ideal of $\mathscr{A}$.

5.2 Main results

Before stating the main results of this paper, we review some concepts needed in the sequel. If $\mathscr{A}$ has a minimal left (or right) ideal, then its *socle*, denoted by $\mathrm{Soc}(\mathscr{A})$, is the sum of all minimal left (or right) ideals of $\mathscr{A}$. If $\mathscr{A}$ has no minimal one-sided ideals, the socle is then defined to be trivial; i.e., $\mathrm{Soc}(\mathscr{A}) = \{0\}$. Note that $\mathrm{Soc}(\mathscr{A})$ is an ideal of $\mathscr{A}$ consisting of all elements $a \in \mathscr{A}$ for which $\sigma(xa)$ is finite for all $x \in \mathscr{A}$, and that all its elements are algebraic. A nonzero element $u \in \mathscr{A}$ is said to have *rank one* if for every $x \in \mathscr{A}$, the spectrum $\sigma(ux)$ contains at most one nonzero scalar. The set $\mathscr{F}_1(\mathscr{A})$ of all rank one elements of $\mathscr{A}$ is contained in $\mathrm{Soc}(\mathscr{A})$ and in turn $\mathrm{Soc}(\mathscr{A})$ is equal to the set of all finite sums of rank one elements of $\mathscr{A}$. An ideal P of $\mathscr{A}$ is *essential* if 0 is the only element a of $\mathscr{A}$ for which $a.P = 0$. We refer the reader to [6, 9] for more details. It should be noted that a semisimple Banach algebra is finite dimensional if and only if it coincides with its socle; see for instance [6, Theorem 5.4.2]. Finally, recall that a linear map φ from $\mathscr{A}$ into $\mathscr{B}$ is said to be *Jordan homomorphism* if $\varphi(a^2) = \varphi(a)^2$ for all $a \in \mathscr{A}$, and that a bijective Jordan homomorphism is called a *Jordan isomorphism*.

The following result gives a complete description of maps between Banach algebras preserving the spectrum of product of elements.

Theorem 5.2.1. *Assume that $\mathscr{A}$ is semisimple and $\mathscr{B}$ has an essential socle. If maps φ_1 and φ_2 from $\mathscr{A}$ onto $\mathscr{B}$ satisfy*

$$\sigma(\varphi_1(a)\varphi_2(b)) = \sigma(ab), \quad (a, b \in \mathscr{A}), \tag{5.2.3}$$

then the maps $\varphi_1\varphi_2(\mathbf{1})$ and $\varphi_1(\mathbf{1})\varphi_2$ coincide and are Jordan isomorphisms.

Clearly, every homomorphism and every anti-homomorphism is a Jordan homomorphism. Conversely, in [51], Jacobson and Rickart proved that a Jordan homomorphism from an arbitrary ring into a domain is either a homomorphism or an antihomomorphism. The same conclusion holds for Jordan homomorphisms onto prime rings, as shown by Herstein [44] and Smiley [78]. In fact, the problem of whether Jordan homomorphisms can be expressed through homomorphisms and antihomomorphisms was considered by several authors; see for instance [10, 22] and the references therein.

The first corollary is an immediate consequence of Theorem 5.2.1 and Herstein's theorem [44].

Corollary 5.2.2. *If $\mathscr{A}$ is a primitive Banach algebra with nonzero socle, and φ_1 and φ_2 are surjective maps from $\mathscr{A}$ into $\mathscr{B}$ satisfying (5.2.3), then the maps $\varphi_1\varphi_2(\mathbf{1})$ and $\varphi_1(\mathbf{1})\varphi_2$ coincide, and $\varphi_1\varphi_2(\mathbf{1})$ is either an isomorphism or an anti-isomorphism.*

Jacobson's lemma [6, Lemma 3.1.2] asserts that for arbitrary $a, b \in \mathscr{A}$, the two spectra $\sigma(ab)$ and $\sigma(ba)$ can differ by only the point 0. With this, it is easy to see that $\sigma(ab) = \sigma(ba)$ for all $a, b \in \mathscr{A}$ if and only if every left/right invertible element in $\mathscr{A}$ is invertible if and only if every anti-isomorphism from $\mathscr{A}$ into $\mathscr{B}$ satisfy (5.1.1). This happens, in particular, when $\mathscr{A}$ and $\mathscr{B}$ coincide with $M_n(\mathbb{C})$, the algebra of all $n \times n$-matrices.

The second corollary is an immediate consequence of Theorem 5.2.1 when φ_1 and φ_2 are the same.

Corollary 5.2.3. *Assume that $\mathscr{A}$ is semisimple and $\mathscr{B}$ has an essential socle. If a map φ from $\mathscr{A}$ onto $\mathscr{B}$ satisfies (5.1.1), then $\varphi(\mathbf{1})$ is a central invertible element of $\mathscr{B}$ for which $\varphi(\mathbf{1})^2 = \mathbf{1}$ and $\varphi(\mathbf{1})\varphi$ is a Jordan isomorphism.*

The final result describes the form of all maps between Banach algebras preserving the spectrum of triple product of elements.

Theorem 5.2.4. *Assume that $\mathscr{A}$ is semisimple and $\mathscr{B}$ has an essential socle. A surjective map $\varphi : \mathscr{A} \to \mathscr{B}$ satisfies (5.1.2) if and only if $\varphi(\mathbf{1})$ is a central invertible element of $\mathscr{B}$ for which $\varphi(\mathbf{1})^3 = \mathbf{1}$ and $\varphi(\mathbf{1})^2\varphi$ is a Jordan isomorphism.*

To prove the above results, our strategy is to use spectral and algebraic methods and show that the hypotheses of Theorem 5.2.1 and Theorem 5.2.4 allow us to exploit the following result of Brešar, Fošner and Šemrl [25] and deduce that the corresponding mappings are Jordan isomorphisms.

Theorem 5.2.5. *[Brešar, Fošner and Šemrl [25]] If $\mathscr{A}$ and $\mathscr{B}$ are semisimple Banach algebras such that Soc($\mathscr{A}$) is essential, and $\varphi : \mathscr{A} \to \mathscr{B}$ is a bijective linear map preserving the invertibility, then φ is a Jordan isomorphism.*

We close this section by pointing out that if $\mathscr{A}$ or $\mathscr{B}$ is isometrically isomorphic to $\mathbb{C}$, then the proof of our main results is an easy exercise. Thus, throughout the rest of this paper, one may assume that each of our algebras $\mathscr{A}$ and $\mathscr{B}$ has a dimension greater than 1.

5.3 Preliminaries and properties of maps preserving the peripheral spectrum

In this section, we collect and establish some auxiliary results needed for the proofs of the main results. Some of this could be of independent interest.

For any element a in $\mathscr{A}$, let $\sigma_\pi(a)$ be the *peripheral spectrum* defined by

$$\sigma_\pi(a) := \{\lambda \in \sigma(a) : |\lambda| = \mathrm{r}(a)\}.$$

It is a nonempty compact subset of $\sigma(a)$. The following lemma is a variant of [21, Theorem 2.2], and its proof uses rank one elements. It is inspired by [9], and uses some tools quoted from [9, 25] where it is shown that there is a function τ on $\mathscr{F}_1(\mathscr{A})$ such that $\sigma(u) = \{0, \tau(u)\}$ for all $u \in \mathscr{F}_1(\mathscr{A})$ and that

$$\tau(au + bu) = \tau(au) + \tau(bu) \tag{5.3.4}$$

for all $a,\ b \in \mathscr{A}$ and $u \in \mathscr{F}_1(\mathscr{A})$.

Lemma 5.3.1. *If $\mathscr{A}$ is semisimple, then, for every two elements $a,\ b \in \mathscr{A}$, the following statements are equivalent.*

(i) $\sigma(au) = \sigma(bu)$ *for all* $u \in \mathscr{F}_1(\mathscr{A})$.

(ii) $\sigma_\pi(au) = \sigma_\pi(bu)$ *for all* $u \in \mathscr{F}_1(\mathscr{A})$.

(iii) $\tau(au) = \tau(bu)$ *for all* $u \in \mathscr{F}_1(\mathscr{A})$.

(iv) $(a - b) \cdot Soc(\mathscr{A}) = \{0\}$. *In particular,* $a = b$ *if* $Soc(\mathscr{A})$ *is essential.*

Proof. We only need to establish the implication (3) $\Rightarrow$ (4). Assume that $\tau(au) = \tau(bu)$ for all $u \in \mathscr{F}_1(\mathscr{A})$. By (5.3.4), we have

$$\tau\big((a - b)u\big) = 0 \tag{5.3.5}$$

for all $u \in \mathscr{F}_1(\mathscr{A})$. Now, assume that $(a-b)u_0 \neq 0$ for some $u_0 \in \mathscr{F}_1(\mathscr{A})$, and note that, since $\mathscr{A}$ is semi simple, there is $x \in \mathscr{A}$ such that $\sigma((a-b)u_0 x) \neq \{0\}$. This implies that $\tau((a-b)u_0 x) \neq 0$ and shows that (5.3.5) does not hold for $u = u_0 x$. This contradiction shows $(a-b)u = 0$ for all $u \in \mathscr{F}_1(\mathscr{A})$, and the implication (3) $\Rightarrow$ (4) is established. □

In the sequel, let φ_1 and φ_2 be two maps from $\mathscr{A}$ onto $\mathscr{B}$ such that

$$\sigma_\pi(\varphi_1(a)\varphi_2(b)) = \sigma_\pi(ab), \quad \text{for all } a, b \in \mathscr{A}. \tag{5.3.6}$$

The following lemma shows that if (5.3.6) is satisfied and $\mathscr{A}$ is semisimple, then so is $\mathscr{B}$. Recall that the Jacobson radical of $\mathscr{A}$, denoted by $\mathrm{rad}(\mathscr{A})$, is the intersection of all maximal left (or right) ideals of $\mathscr{A}$ and coincides with the collection of all elements $x \in \mathscr{A}$ satisfying $\mathrm{r}(xa) = 0$ for all $a \in \mathscr{A}$.

Lemma 5.3.2. *If φ_1 and φ_2 are maps from $\mathscr{A}$ onto $\mathscr{B}$ satisfying (5.3.6), then the following assertions hold.*

(i) $\varphi_1(rad(\mathscr{A})) = rad(\mathscr{B})$ and $\varphi_2(rad(\mathscr{A})) = rad(\mathscr{B})$.

(ii) If $\mathscr{A}$ is semisimple, then so is $\mathscr{B}$.

(iii) If $\mathscr{A}$ is semisimple, then 0 is the only element $a \in \mathscr{A}$ for which $\varphi_1(a) = 0$ or $\varphi_2(a) = 0$.

Proof. (i) Assume that $a \in \mathrm{rad}(\mathscr{A})$ and note that, since φ_1 and φ_2 satisfy (5.3.6), we have

$$\{0\} = \sigma(ax) \Rightarrow \{0\} = \sigma_\pi(ax) = \sigma_\pi\big(\varphi_1(a)\varphi_2(x)\big) \Rightarrow \sigma\big(\varphi_1(a)\varphi_2(x)\big) = \{0\}$$

for all $x \in \mathscr{A}$. This and the surjectivity of φ_2 show that $\mathrm{r}\big(\varphi_1(a)y\big) = 0$ for all $y \in \mathscr{B}$. Hence $\varphi_1(a) \in \mathrm{rad}(\mathscr{B})$ and thus $\varphi_1(\mathrm{rad}(\mathscr{A})) \subset \mathrm{rad}(\mathscr{B})$. Since φ_1 is surjective, the reverse inclusion holds in a similar way.

(ii) If $\mathscr{A}$ is semisimple, then $\mathrm{rad}(\mathscr{A}) = \{0\}$ and thus $\mathrm{rad}(\mathscr{B}) = \{\varphi_1(0)\}$ implies that $\varphi_1(0) = 0$ and hence $\mathscr{B}$ is also semisimple.

(iii) If $\mathscr{A}$ is semisimple and $\varphi_1(a) = 0$ for some $a \in \mathscr{A}$, then

$$\{0\} = \sigma(0 \times \varphi_2(x)) = \sigma(\varphi_1(a)\varphi_2(x)) \Rightarrow \{0\} = \sigma_\pi(\varphi_1(a)\varphi_2(x)) = \sigma_\pi(ax) \Rightarrow \{0\} = \sigma(ax)$$

for all $x \in \mathscr{A}$, and thus $a = 0$ since $\mathscr{A}$ is semisimple. The statements for φ_2 can be shown in a similar way. □

We also need the next lemma as a tool.

Lemma 5.3.3. *If K is a compact subset of the open unit disc and contains at least two nonzero elements, then there is a rational function f with the following properties:*

(i) $f(0) = 0$.

(ii) $\deg(f) \le 2$*, and thus f has the form*

$$f(z) = \frac{Az^2 + Bz}{Cz^2 + Dz + E}$$

for some suitable constants $A, \dots, E$.

(iii) *f is analytic on an open neighborhood of K.*

(iv) *$f(K)$ is a subset of closed unit disc that intersects the unit circle at two or more points.*

Proof. We may assume that $(0,1) \cap K \neq \emptyset$ and also $(-1,0) \cap K \neq \emptyset$. If this is not the case, since there are two distinct points $z_1, z_2 \in K \setminus \{0\}$, we can apply $h(z) = az + bz^2$ with appropriate coefficients a and b such that $h(z_1) \in (0,1)$ and $h(z_2) \in (-1,0)$ and moreover $h(K)$ still stays inside the open unit disc $\mathbb{D}$. For example, with $t > 0$ and

$$a = -t\frac{z_1^2 + z_2^2}{z_1 z_2 (z_1 - z_2)} \qquad \text{and} \qquad b = t\frac{z_1 + z_2}{z_1 z_2 (z_1 - z_2)},$$

we get $h(z_1) = t$ and $h(z_2) = -t$ and then with t small enough $h(K)$ resides in $\mathbb{D}$.

Since $h(K)$ has at least two nonzero elements, there is a disc $D(c,r)$ centered at point c and of radius r such that $h(K) \subset \overline{D(c,r)}$ and moreover $h(K)$ intersects $\partial D(c,r)$ at least at two distinct points. The verification is geometrical as follows. Consider all closed discs $\overline{D(c,r)}$ which contains $h(K)$ and let $r_0 = \inf r$. Hence, there is a sequence (c_n, r_n) such that $h(K) \subset \overline{D(c_n, r_n)}$ and $r_n \to r_0$. Without loss of generality, we may also assume that c_n is convergent, say to c_0, since otherwise we can take a subsequence with such a property. Therefore, $h(K) \subset \overline{D(c_0, r_0)}$ and r_0 is the smallest radius for which the preceding inclusion works. Note that since t and $-t$ are in K, we must have $r_0 > 0$ and $0 \in D(c_0, r_0)$. The disc $\overline{D(c_0, r_0)}$ fulfills the required property, i.e., $h(K)$ touches its frontiers at least at two distinct points. First of all, $h(K)$ has to touch the frontier at a point since otherwise we can decrease r_0. Secondly, if it touches just at one point, say a, we can translate the disc on the direction of diagonal passing through a by $\epsilon > 0$, where ε is small enough, and put $h(K)$ inside the translated disc. Hence, its radius can again be decreased, which is a contradiction. Therefore, $h(K)$ must touch $\partial D(c_0, r_0)$ at least at two points.

Put

$$g(z) = (c - z)/r.$$

Then $g \circ h$ fulfils the properties $(2) - (3) - (4)$ except that possibly $(g \circ h)(0) = c/r \neq 0$. To arrange for this property, if it does not hold, we are forced to introduce a finite pole. Consider the disc automorphism

$$\tau(z) = \frac{d - z}{1 - \bar{d}z},$$

where $d = c/r \in \mathbb{D}$. Then the function

$$f(z) = (\tau \circ g \circ h)(z) = \frac{r(az + bz^2)}{\bar{c}(az + bz^2) + r^2 - |c|^2}$$

satisfies all the properties $(1) - (4)$. □

The next lemma shows that if φ_1 and φ_2 are maps from $\mathscr{A}$ onto $\mathscr{B}$ satisfying (5.3.6) and $\mathscr{A}$ is semisimple, then either both socles of $\mathscr{A}$ and $\mathscr{B}$ are essential or both are not. It also tells us that both maps preserve the rank one elements in both directions. Its proof uses the above lemma.

Lemma 5.3.4. *Assume that $\mathscr{A}$ is semisimple, and φ_1 and φ_2 are maps from $\mathscr{A}$ onto $\mathscr{B}$ satisfying (5.3.6). Then the following statements hold.*

(i) Both maps φ_1 and φ_2 preserve rank one elements; i.e., $\varphi_1\big(\mathscr{F}_1(\mathscr{A})\big) = \mathscr{F}_1(\mathscr{B})$ and $\varphi_2\big(\mathscr{F}_1(\mathscr{A})\big) = \mathscr{F}_1(\mathscr{B})$.

(ii) Soc$(\mathscr{A})$ is essential if and only if Soc$(\mathscr{B})$ is essential.

Proof. (i) Let $u \in \mathscr{A}$ be a rank one element, and note that $\varphi_1(u) \neq 0$, by Lemma 5.3.2. Our aim is to show that $\varphi_1(u)$ is a rank one element in $\mathscr{B}$ by proving that $\sigma(\varphi_1(u)y)$ contains at most one nonzero element for all $y \in \mathscr{B}$. By (5.3.6), we have

$$\sigma_\pi(\varphi_1(u)\varphi_2(x)) = \sigma_\pi(ux) = \{\tau(ux)\}$$

for all $x \in \mathscr{A}$. As φ_2 is surjective, we see that $\sigma_\pi(\varphi_1(u)y)$ contains one element for all $y \in \mathscr{B}$. Now, for the sake of contradiction, suppose that there exists $y_0 \in \mathscr{B}$ such that $\sigma(\varphi_1(u)y_0)$ contains at least two nonzero elements. Without loss of generality, we may assume that spectral radius of $\varphi_1(u)y_0$ is less than $1/2$. By Lemma 5.3.3, we conclude that there exists an analytic function f on the open unit disc $\mathbb{D}$ such that $f(0) = 0$ and $f(\sigma(\varphi_1(u)y_0))$ is a subset of closed unit disc that intersects the unit circle $\mathbb{T}$ at two or more points. Moreover, the spectral mapping theorem tells us that

$$f(\sigma(\varphi_1(u)y_0)) = \sigma(f(\varphi_1(u)y_0)). \tag{5.3.7}$$

As f is analytic on $\mathbb{D}$ and $f(0) = 0$, we can write $f(z) = zg(z)$, $(z \in \mathbb{D})$, where g is also analytic on $\mathbb{D}$. From (5.3.7), the spectrum of the element $f(\varphi_1(u)y_0) = \varphi_1(u)y_0 g(\varphi_1(u)y_0)$ is a subset of closed unit disk and intersects $\mathbb{T}$ at two or more points. This contradicts the fact that $\sigma_\pi(\varphi_1(u)y)$ contains one element for all $y \in \mathscr{B}$, and shows that $\varphi_1(u)$ is a rank one element in $\mathscr{B}$; as desired.

(ii) Assume that Soc$(\mathscr{B})$ is essential and let $x \in \mathscr{A}$ such that $x.$Soc$(\mathscr{A}) = \{0\}$. In particular, we have $xu = 0$ and thus

$$\{0\} = \sigma(xu) \;\Rightarrow\; \{0\} = \sigma_\pi(xu) = \sigma_\pi(\varphi_1(x)\varphi_2(u)) \;\Rightarrow\; \{0\} = \sigma(\varphi_1(x)\varphi_2(u))$$

for all $u \in \mathscr{F}_1(\mathscr{A})$. From this and (1), it follows that $\tau(\varphi_1(x)v) = 0$ for all $v \in \mathscr{F}_1(\mathscr{B})$. Note that, since $\mathscr{B}$ is semisimple too by Lemma 5.3.2, the essentiality of the socle of $\mathscr{B}$ and Lemma 5.3.1 show that $\varphi_1(x) = 0$. Now, Lemma 5.3.2 tells us that $x = 0$ and Soc$(\mathscr{A})$ is essential. The converse runs in a similar way and we therefore omit the details. □

The next lemma is a variant of Lemma 5.3.1, and is needed for the proof of Theorem 5.2.4. Its proof uses Lemma 5.3.1.

Lemma 5.3.5. *For every two elements a and b in a semisimple Banach algebra $\mathscr{A}$, the following statements are equivalent.*

(i) $\sigma(uau) = \sigma(ubu)$ *for all* $u \in \mathscr{F}_1(\mathscr{A})$.

(ii) $\sigma_\pi(uau) = \sigma_\pi(ubu)$ *for all* $u \in \mathscr{F}_1(\mathscr{A})$.

(iii) $\tau(uau) = \tau(ubu)$ *for all* $u \in \mathscr{F}_1(\mathscr{A})$.

(iv) $\tau(uau) = \tau(ubu)$ *for all nonquasinilpotent elements* $u \in \mathscr{F}_1(\mathscr{A})$.

(v) $(a-b)\cdot Soc(\mathscr{A}) = \{0\}$. *In particular,* $a = b$ *if* $Soc(\mathscr{A})$ *is essential.*

Proof. We only need to show that the implication $(iv) \Rightarrow (v)$ holds. Assume that $\tau(uau) = \tau(ubu)$ for all nonquasinilpotent elements $u \in \mathscr{F}_1(\mathscr{A})$, and note that, since $\tau(au)\tau(u) = \tau(uau) = \tau(ubu) = \tau(bu)\tau(u)$, we have

$$\tau(au) = \tau(bu) \tag{5.3.8}$$

for all nonquasinilpotent elements $u \in \mathscr{F}_1(\mathscr{A})$. Now, we shall show that (5.3.8) holds for all $u \in \mathscr{F}_1(\mathscr{A})$. Fix a quasinilpotent element $u \in \mathscr{F}_1(\mathscr{A})$ so that $\tau(u) = 0$, and note that, since $\mathscr{A}$ is semisimple, there is $x \in \mathscr{A}$ such that $\tau(xu) \neq 0$ and thus $\tau((\mathbf{1}-x)u) = -\tau(xu) \neq 0$. Therefore, (5.3.8) applied to the nonquasinilpotent elements xu and $(\mathbf{1}-x)u$ gives that

$$\tau(au) = \tau(axu) + \tau(a(\mathbf{1}-x)u) = \tau(bxu) + \tau(b(\mathbf{1}-x)u) = \tau(bu).$$

Now, we have $\tau(au) = \tau(bu)$ for all $u \in \mathscr{F}_1(\mathscr{A})$ and Lemma 5.3.1 implies that $(a-b)\cdot \mathrm{Soc}(\mathscr{A}) = \{0\}$; as desired. $\square$

Now, we turn our attention to the permanence properties of maps between Banach algebras preserving the peripheral spectrum of triple products of elements. In what follows, let φ_1 and φ_2 be two maps from $\mathscr{A}$ onto $\mathscr{B}$ such that

$$\sigma_\pi(\varphi_1(a)\varphi_2(b)\varphi_1(a)) = \sigma_\pi(aba), \quad \text{for all } a, b \in \mathscr{A}. \tag{5.3.9}$$

The following lemma shows that if (5.3.9) is satisfied and $\mathscr{A}$ is semisimple, then so is $\mathscr{B}$. It is a variant of Lemma 5.3.2.

Lemma 5.3.6. *If φ_1 and φ_2 are maps from $\mathscr{A}$ onto $\mathscr{B}$ satisfying (5.3.9) and if $\mathscr{A}$ is semisimple, then $\mathscr{B}$ is semisimple.*

Proof. Let $b \in \mathrm{rad}(\mathscr{B})$, and note that, since φ_2 is surjective, there is $a \in \mathscr{A}$ such that $\varphi_2(a) = b$. For every $x \in \mathscr{A}$, we have $\varphi_1(x)b\varphi_1(x) \in \mathrm{rad}(\mathscr{B})$ and thus

$$\{0\} = \sigma_\pi\big(\varphi_1(x)b\varphi_1(x)\big) = \sigma_\pi\big(\varphi_1(x)\varphi_2(a)\varphi_1(x)\big) = \sigma_\pi(xax).$$

As $\mathscr{A}$ is semisimple, Theorem 5.6.3 implies that $a = 0$, and thus $\mathrm{rad}(\mathscr{B}) \subset \{\varphi_2(0)\}$. As $\mathrm{rad}(\mathscr{B})$ is an ideal, we conclude that $\mathrm{rad}(\mathscr{B}) = \{0\}$ and $\mathscr{B}$ is semisimple. $\square$

5.4 Proof of Theorem 5.2.1

The proof consists of several steps.

Step 1. Both φ_1 and φ_2 are linear and bijective.

By Lemma 5.3.2 and Lemma 5.3.4, both algebras $\mathscr{A}$ and $\mathscr{B}$ are semisimple with essential socles, and both maps φ_1 and φ_2 preserve elements of rank one. Now, we proceed to show first that φ_1 and φ_2 are additive. Let $v \in \mathscr{B}$ be a rank one element of $\mathscr{B}$, and let $u \in \mathscr{A}$ be a rank one element such that $\varphi_2(u) = v$. Since φ_1 and φ_2 satisfy (5.2.3), we have

$$\begin{aligned}\tau\big(\varphi_1(a+b)v\big) &= \tau(\varphi_1(a+b)\varphi_2(u))\\ &= \tau((a+b)u) = \tau(au) + \tau(bu)\\ &= \tau(\varphi_1(a)\varphi_2(u)) + \tau(\varphi_1(b)\varphi_2(u))\\ &= \tau\big(\big(\varphi_1(a)+\varphi_1(b)\big)\varphi_2(u)\big)\\ &= \tau\big(\big(\varphi_1(a)+\varphi_1(b)\big)v\big)\end{aligned}$$

for all $a,\ b \in \mathscr{A}$. By the arbitrariness of the rank one element $v \in \mathscr{B}$ and Lemma 5.3.1, we conclude that

$$\varphi_1(a+b) = \varphi_1(a) + \varphi_1(b)$$

for all $a,\ b \in \mathscr{A}$, and so φ_1 is additive. The additivity of φ_2 can be shown in a similar way.

Next, we show that φ_1 and φ_2 are homogenous. For every $a \in \mathscr{A}$ and $\lambda \in \mathbb{C}$, we have

$$\sigma(\lambda\varphi_1(a)\varphi_2(u)) = \lambda\sigma(\varphi_1(a)\varphi_2(u)) = \lambda\sigma(au) = \sigma(\lambda au) = \sigma(\varphi_1(\lambda a)\varphi_2(u))$$

for all $u \in \mathscr{F}_1(\mathscr{A})$. By Lemma 5.3.1, we have $\varphi_1(\lambda a) = \lambda\varphi_1(a)$ for all $a \in \mathscr{A}$ and $\lambda \in \mathbb{C}$, and φ_1 is homogenous. The homogeneity of φ_2 can be shown in a similar way.

As φ_1 and φ_2 are linear, Lemma 5.3.2 tells us that φ_1 and φ_2 are injective and are, in fact, bijective.

Step 2. $\varphi_1(\mathbf{1})\varphi_2$ and $\varphi_1\varphi_2(\mathbf{1})$ are Jordan isomorphisms.

First we show that $\varphi_1(\mathbf{1})\varphi_2$ is surjective. Let $z \in \mathscr{B}$, and let us show that there exists $x \in \mathscr{A}$ such that $\varphi_1(\mathbf{1})\varphi_2(x) = z$. By (5.2.3), we have $\sigma(\varphi_1(\mathbf{1})\varphi_2(\mathbf{1})) = \{1\}$ and $\varphi_1(\mathbf{1})\varphi_2(\mathbf{1})$ is an invertible element. So, there exists $y \in \mathscr{B}$ such that $\varphi_1(\mathbf{1})\varphi_2(\mathbf{1})y = \mathbf{1}$. This multiplied by z gives $\varphi_1(\mathbf{1})\varphi_2(\mathbf{1})yz = z$. As φ_2 is surjective, there exists $x \in \mathscr{A}$ such that $\varphi_2(x) = \varphi_2(\mathbf{1})yz$ and thus

$$\varphi_1(\mathbf{1})\varphi_2(x) = \varphi_1(\mathbf{1})\varphi_2(\mathbf{1})yz = z.$$

This shows that $\varphi_1(\mathbf{1})\varphi_2(x)$ is surjective; as claimed.

From (5.2.3) we have

$$\sigma\big(\varphi_1(\mathbf{1})\varphi_2(x)\big) = \sigma(x)$$

for all $x \in \mathscr{A}$. As $\mathscr{A}$ is semisimple and $\varphi_1(1)\varphi_2$ is a linear surjective map that preserves the spectrum, we conclude that $\varphi_1(\mathbf{1})\varphi_2$ is a Jordan isomorphism; see Theorem 5.2.5.

By the same way we can show that $\varphi_1\varphi_2(\mathbf{1})$ is a Jordan isomorphism.

Step 3. Both elements $\varphi_1(\mathbf{1})$ and $\varphi_2(\mathbf{1})$ are invertible, and

$$\varphi_1(\mathbf{1})\varphi_2(\mathbf{1}) = \varphi_2(\mathbf{1})\varphi_1(\mathbf{1}) = \mathbf{1}. \tag{5.4.10}$$

Since $\varphi_1\varphi_2(\mathbf{1})$ is a Jordan isomorphism, we see that $\varphi_1(\mathbf{1})\varphi_2(\mathbf{1}) = \mathbf{1}$. So, we only need to show that $\varphi_2(\mathbf{1})\varphi_1(\mathbf{1}) = \mathbf{1}$. As φ_2 is surjective, there exists x such that $\varphi_2(x) = \mathbf{1} - \varphi_2(\mathbf{1})\varphi_1(\mathbf{1})$. We have

$$\varphi_1(\mathbf{1})\varphi_2(x) = \varphi_1(\mathbf{1})\big(1 - \varphi_2(\mathbf{1})\varphi_1(\mathbf{1})\big) = \varphi_1(\mathbf{1}) - \varphi_1(\mathbf{1})\varphi_2(\mathbf{1})\varphi_1(\mathbf{1}) = 0.$$

As $\varphi_1(\mathbf{1})\varphi_2$ is bijective, we see that $x = 0$ and $\mathbf{1} - \varphi_2(\mathbf{1})\varphi_1(\mathbf{1}) = \varphi_2(x) = \varphi_2(0) = 0$; as desired.

Step 4. $\varphi_1(\mathbf{1})\varphi_2 = \varphi_1\varphi_2(\mathbf{1})$.

Indeed, let $a \in \mathscr{A}$ be an invertible element and let $\psi_1(x) := \varphi_1(xa)\varphi_2(\mathbf{1})$ and $\psi_2(x) := \varphi_1(\mathbf{1})\varphi_2(a^{-1}x)$ for all $x \in \mathscr{A}$. Clearly, we have

$$\sigma(\psi_1(x)\psi_2(y)) = \sigma(xy)$$

for all $x,\ y \in \mathscr{A}$. By Step 3, we have $\psi_1(\mathbf{1}) = \{\psi_2(\mathbf{1})\}^{-1}$ and thus

$$\varphi_1(a)\varphi_2(\mathbf{1}) = \{\varphi_1(\mathbf{1})\varphi_2(a^{-1})\}^{-1} = \varphi_1(\mathbf{1})\varphi_2(a)$$

since $\varphi_1(\mathbf{1})\varphi_2$ is a Jordan isomorphism. From this and the linearity of both maps $\varphi_1\varphi_2(\mathbf{1})$ and $\varphi_1(\mathbf{1})\varphi_2$, we infer that this identity holds, in fact, for any arbitrary $a \in \mathscr{A}$.

5.5 Proof of Theorem 5.2.4

The "if" part is obvious since any Jordan isomorphism ϕ between $\mathscr{A}$ and $\mathscr{B}$ preserves the spectrum and satisfies $\phi(xyx) = \phi(x)\phi(y)\phi(x)$ for all $x, y \in \mathscr{A}$.

Conversely, assume that $\mathscr{A}$ is semisimple and $\mathscr{B}$ has an essential socle, and φ is a map from $\mathscr{A}$ onto $\mathscr{B}$ satisfying (5.1.2). Note that, by Lemma 5.3.6, $\mathscr{B}$ is semisimple too. Now, let us first prove that φ is a linear map. To do so, let us first show that φ is homogenous. Indeed, for every $a,\ b \in \mathscr{A}$ and $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}\sigma\big(\varphi(a)\lambda\varphi(b)\varphi(a)\big) &= \lambda\sigma\big(\varphi(a)\varphi(b)\varphi(a)\big) = \lambda\sigma(aba) = \sigma(a\lambda ba) = \\ &= \sigma\big(\varphi(a)\varphi(\lambda b)\varphi(a)\big).\end{aligned}$$

As φ is surjective, Lemma 5.3.5 tell us that $\lambda\varphi(b) = \varphi(\lambda b)$ for all $b \in \mathscr{A}$ and $\lambda \in \mathbb{C}$, and thus φ is homogenous.

Second, let us show that φ is additive. From (5.1.2) and Jacobson's lemma, we have

$$\sigma\big(\varphi(x)(\varphi(u))^2\big) \cup \{0\} = \sigma(xu^2) \cup \{0\} \qquad (x, u \in \mathscr{A}).$$

Taking into consideration that φ is surjective, we conclude that $u^2 \in \mathscr{F}_1(\mathscr{A})$ if and only if $\big(\varphi(u)\big)^2 \in \mathscr{F}_1(\mathscr{B}))$. By taking $x = a + b$ and $u \in \mathscr{A}$ such that $u^2 \in \mathscr{F}_1(\mathscr{A})$, we obtain

$$\sigma\big(\varphi(a+b)(\varphi(u))^2\big) \cup \{0\} = \sigma((a+b)u^2) \cup \{0\} \qquad (a,\ b \in \mathscr{A}).$$

Just as in the proof of Theorem 5.2.1, we get

$$\tau\big(v\varphi(a+b)v\big) = \tau\big(\varphi(a+b)v^2\big) = \tau\big(\big(\varphi(a)+\varphi(b)\big)v^2\big) = \tau\big(v\big(\varphi(a)+\varphi(b)\big)v\big)$$

for all $v \in \mathscr{F}_1(\mathscr{B})$. Lemma 5.3.5 implies that

$$\varphi(a+b) = \varphi(a) + \varphi(b) \qquad (a, b \in \mathscr{A}),$$

and φ is additive; as claimed.

Now, consider the linear map $\phi := \varphi(\mathbf{1})\varphi(.)\varphi(\mathbf{1})$, and note that, by (5.1.2), it satisfies

$$\sigma(x) = \sigma(\phi(x)) \qquad (x \in \mathscr{A}).$$

As $\{1\} = \sigma(\varphi(\mathbf{1})\varphi(\mathbf{1})\varphi(\mathbf{1}))$, we see that $\varphi(\mathbf{1})$ is invertible and thus ϕ is a surjective map. By Theorem 5.2.5, we have that $\phi = \varphi(\mathbf{1})\varphi\varphi(\mathbf{1})$ is a Jordan isomorphism. Consequently, we have

$$\phi(\mathbf{1}) = \varphi(\mathbf{1})\varphi(\mathbf{1})\varphi(\mathbf{1}) = \mathbf{1}$$

and that $\varphi(\mathbf{1})$ is an invertible element. To finish, we only need to show that $\varphi(\mathbf{1})$ is a central element. From (5.1.2) we have $\sigma\big(\varphi(\mathbf{1})\varphi(a)\varphi(\mathbf{1})\big)^3 = \sigma(a)^3 = \sigma(a^3) = \sigma(\varphi(a)^3) = \sigma(\varphi(a))^3$ for all $a \in \mathscr{A}$. It follows that $\mathrm{r}(\varphi(\mathbf{1})^2\varphi(a)) = \mathrm{r}(\varphi(a))$ for all $a \in \mathscr{A}$ and thus, by [24, Theorem 3.1], $\varphi(\mathbf{1})^2 = \varphi(\mathbf{1})^{-1}$ is a central element of $\mathscr{B}$.

5.6 Remarks and comments

In Section 3, we established some permanence properties of maps preserving the peripheral spectrum and we could describe the form of such maps if the algebras $\mathscr{A}$ and $\mathscr{B}$ are of particular type such as being von Neumann algebras. We also would like to point out that we can particularize Theorem 5.2.1 in the case of the algebra of operators on a complex Banach space, and obtain the main result of [68, 69]. Let X and Y denote infinite-dimensional complex Banach spaces and $\mathscr{B}(X, Y)$ denote the space of all bounded linear maps from X into Y. When $X = Y$, we simply write $\mathscr{B}(X)$ instead of $\mathscr{B}(X, X)$. For any operator $T \in \mathscr{B}(X)$, let T^* denote as usual its adjoint on the dual space X^* of X.

Corollary 5.6.1. *Two maps φ_1 and φ_2 from $\mathscr{B}(X)$ onto $\mathscr{B}(Y)$ satisfy (5.2.3) if and only if one of the following statements holds.*

(i) There exist two bijective mappings $A,\ B \in \mathscr{B}(X,Y)$ such that $\varphi_1(T) = ATB^{-1}$ and $\varphi_2(T) = BTA^{-1}$ for all $T \in \mathscr{B}(X)$.

(ii) There exist two bijective mappings $A,\ B \in \mathscr{B}(X^*,Y)$ such that $\varphi_1(T) = AT^*B^{-1}$ and $\varphi_2(T) = BT^*A^{-1}$ for all $T \in \mathscr{B}(X)$. This case can not occur if X or Y is not reflexive.

Recently, Argyros and Haydon in [4] constructed a complex separable and infinite-dimensional Banach space X such that $\mathscr{B}(X) = \mathbb{C}\mathbf{1} + \mathscr{K}(X)$, where $\mathscr{K}(X)$ is the closed ideal of all compact operators on X. For such a space X, every left/right invertible operator $T \in \mathscr{B}(X)$ is invertible and thus $\sigma(TS) = \sigma(ST)$ for all $T, S \in \mathscr{B}(X)$. So, the map φ from $\mathscr{B}(X)$ to $\mathscr{B}(X^*)$ defined by $\varphi(T) := \pm T^*$ for all $T \in \mathscr{B}(X)$ satisfies

$$\sigma(\varphi(T)\varphi(S)) = \sigma(T^*S^*) = \sigma((ST)^*) = \sigma(ST) = \sigma(TS)$$

for all $T,\ S \in \mathscr{B}(X)$. This example shows that a map satisfying (5.1.1) is not necessarily multiplicative.

One can particularize as well Theorem 5.2.4 in the case of the algebra of operators on a complex Banach space. However, we omit the details and leave them for the reader.

In [3, 24], the question whether for given two elements $a,\ b \in \mathscr{A}$ the condition $\sigma(ax) = \sigma(bx)$ for all $x \in \mathscr{A}$ entails $a = b$ was studied, and an affirmative answer has been obtained in [3, 24] for various special cases and for some classes of algebras, including C^*-algebras. While in [21], Braatvedt and Brits showed that the answer is always affirmative if $\mathscr{A}$ is a semismple Banach algebra.

Theorem 5.6.2 ([21])**.** *Let $\mathscr{A}$ be a semisimple Banach algebra and a, b be two elements in $\mathscr{A}$. Then the following assertions are equivalent.*

(i) $a = b$.

(ii) $\sigma(ax) = \sigma(bx)$ for all $x \in \mathscr{A}$.

(iii) $\sigma(ax) = \sigma(bx)$ for all $x \in \mathscr{A}$ satisfying $r(x - \mathbf{1}) < 1$.

The following theorem is variant of Braatvedt and Brits' result that uses the triple product. Its proof is similar to Braatvedt and Brits' result with some improvements for this case.

Theorem 5.6.3. *Let $\mathscr{A}$ be a semisimple Banach algebra and a, b be two elements in $\mathscr{A}$. If*

$$\sigma(xax) = \sigma(xbx) \tag{5.6.11}$$

for all $x \in \mathscr{A}$, then $a = b$.

Proof. Let x be an element in $\mathscr{A}$ satisfying $\mathrm{r}(x-\mathbf{1})<1$, and note that, by the holomorphic calculus, $x=e^y$ for some $y\in\mathscr{A}$. Using (5.6.11), we have

$$\sigma\left(e^{\frac{y}{2}}ae^{\frac{y}{2}}\right)=\sigma\left(e^{\frac{y}{2}}be^{\frac{y}{2}}\right).$$

By Jacobson's lemma, we then have $\sigma(ax)\setminus\{0\}=\sigma(bx)\setminus\{0\}$. But, since x is invertible, we actually obtain $\sigma(ax)=\sigma(bx)$. Since $x\in\mathscr{A}$ is an arbitrary element satisfying $\mathrm{r}(x-\mathbf{1})<1$, Theorem 5.6.2 implies that $a=b$, and the proof is therefore complete. □

In [24], Brešar and Špenko discussed surjective maps between $\mathscr{A}$ and $\mathscr{B}$ satisfying

$$\sigma\big(\varphi(x)\varphi(y)\varphi(z)\big)=\sigma(xyz) \tag{5.6.12}$$

for all $x,y,z\in\mathscr{A}$. The next result shows that such a map is multiplicative up to a multiplicative invertible central element of $\mathscr{B}$. Its proof uses Theorem 5.6.2 and the same arguments provided in [24].

Proposition 5.6.4. *Assume that $\mathscr{A}$ and $\mathscr{B}$ are semisimple Banach algebras. A surjective map $\varphi:\mathscr{A}\to\mathscr{B}$ satisfies (5.6.12) if and only if $\varphi(\mathbf{1})$ is a central invertible element of $\mathscr{B}$ for which $\varphi(\mathbf{1})^3=\mathbf{1}$ and $\varphi(\mathbf{1})^2\varphi$ is isomorphism.*

Proof. Assume that φ satisfies (5.6.12). For every $x,y,z\in\mathscr{A}$, we have

$$\sigma(\varphi(\mathbf{1})\varphi(xy)\varphi(z))=\sigma(xyz)=\sigma(\varphi(x)\varphi(y)\varphi(z)).$$

By the surjectivity of φ and Theorem 5.6.2, we get that $\varphi(\mathbf{1})\varphi(xy)=\varphi(x)\varphi(y)$ for all $x,\ y\in\mathscr{A}$. For $y=\mathbf{1}$, we have $\varphi(\mathbf{1})\varphi(x)=\varphi(x)\varphi(\mathbf{1})$ for all $x\in\mathscr{A}$ and $\varphi(\mathbf{1})$ is a central element of $\mathscr{B}$ for which $\sigma(\varphi(\mathbf{1})^3)=\{1\}$. Hence, $\varphi(\mathbf{1})^3=\mathbf{1}$ and $\varphi(\mathbf{1})^2\varphi$ is multiplicative. □

Conclusion

In this thesis, some nonlinear preserver problems have been considered. In Chapter 3, the characterization of surjective maps $\varphi : \mathscr{B}(X) \to \mathscr{B}(Y)$ that satisfy

$$\mathrm{c}(\varphi(S) \pm \varphi(T)) = \mathrm{c}(S \pm T), \qquad (S, T \in \mathscr{B}(X)),$$

is given, where $\mathrm{c}(\cdot)$ stands either for the minimum modulus, or the surjectivity modulus, or the maximum modulus. The next step could be to extend these results to the general case, when $\mathscr{A}$ and $\mathscr{B}$ are semisimple Banach algebras, i.e., to describe surjective maps $\varphi : \mathscr{A} \to \mathscr{B}$ that satisfy

$$\mathrm{c}(\varphi(a) \pm \varphi(b)) = \mathrm{c}(a \pm b), \qquad (a, b \in \mathscr{A}).$$

In Chapter 4, the characterization of bijective bicontinuous maps $\varphi : \mathscr{B}(X) \to \mathscr{B}(Y)$ that satisfy

$$\gamma(\varphi(S \pm \varphi(T)) = \gamma(S \pm T), \qquad (S, T \in \mathscr{B}(X)),$$

is obtained. The natural extension would be to get rid of the condition of bicontinuity of φ. Also, as in the previous case, it would be good to solve the problem for general semisimple Banach algebras $\mathscr{A}$ and $\mathscr{B}$, i.e., to describe surjective maps $\varphi : \mathscr{A} \to \mathscr{B}$ that satisfy

$$\gamma(\varphi(a) \pm \varphi(b)) = \gamma(a \pm b), \qquad (a, b \in \mathscr{A}).$$

It would be interesting to consider the problems of characterizing surjective maps $\varphi : \mathscr{A} \to \mathscr{B}$ that satisfy one of the following conditions

$$\mathrm{d}(\varphi(a)\varphi(b)) = \mathrm{d}(ab), \qquad (a, b \in \mathscr{A}),$$

or

$$\mathrm{d}(\varphi(a)\varphi(b)\varphi(a)) = \mathrm{d}(aba), \qquad (a, b \in \mathscr{A}),$$

or

$$\mathrm{d}(\varphi(a) \circ \varphi(b)) = \mathrm{d}(a \circ b), \qquad (a, b \in \mathscr{A}),$$

where $\mathrm{d}(\cdot)$ stands either for the minimum modulus, or the surjectivity modulus, or the maximum modulus, or the reduced minimum modulus and in the last equality the Jordan product is considered. Before passing to the general case, one could try to solve these problems for C^*-algebras.

In Chapter 5, the characterization of surjective maps φ_1, φ_2 between semisimple Banach algebra $\mathscr{A}$ and a Banach algebra $\mathscr{B}$ with an essential socle that satisfy

$$\sigma\big(\varphi_1(a)\varphi_2(b)\big) = \sigma(ab), \qquad (a, b \in \mathscr{A}),$$

is given. A next step could be finding similar results for peripheral spectrum, i.e., to describe surjective maps φ_1, φ_2 between semisimple Banach algebra $\mathscr{A}$ and a Banach algebra $\mathscr{B}$ with an essential socle, such that

$$\sigma_\pi\big(\varphi_1(a)\varphi_2(b)\big) = \sigma_\pi(ab), \qquad (a, b \in \mathscr{A}).$$

The condition could be changed to the Jordan product, as follows :

$$\sigma_\pi\big(\varphi_1(a) \circ \varphi_2(b)\big) = \sigma_\pi(a \circ b), \qquad (a, b \in \mathscr{A}).$$

Under the same assumptions on Banach algebras $\mathscr{A}$ and $\mathscr{B}$ the characterization of a surjective map $\varphi : \mathscr{A} \to \mathscr{B}$ that satisfies

$$\sigma\big(\varphi(a)\varphi(b)\varphi(a)\big) = \sigma(aba), \qquad (a, b \in \mathscr{A}),$$

is given. It would be interesting to extend this result to the more general case of surjective maps φ_1 and φ_2 between the Banach algebras $\mathscr{A}$ and $\mathscr{B}$ that satisfy

$$\sigma\big(\varphi_1(a)\varphi_2(b)\varphi_1(a)\big) = \sigma(aba), \qquad (a, b \in \mathscr{A}).$$

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